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Baryon-baryon Couplings in the $^3P_0$ and $^3S_1$ QPC-models

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I. ABSTRACT

In this note we describe the nucleon-nucleon (NN) and hyperon-nucleon (YN) coupling constants in the $^3P_0$- and $^3S_1$-quark-pair-creation (QPC) models. The method used exploits the Fierz-identities, in contrast to papers in the literature. Technically, this means that the treatment of the two QCP-models is rather uniform, and the flavor-spin-color recouplings are rather simple. The description of the meson-states is based on the Van Royen-Weisskopf representation.
We consider the contributions from the 'direct' (a) and the 'exchange' (b) QPC-process. In (a) and (b) the 'active' quark in the initial baryon ends up in the meson, respectively the final baryon. It turns out that (b) is small w.r.t. (a), and neglecting (b) the difference between the $^3P_0$ and $^3S_1$-model is, apart from an overall constant, only due to the different coefficients in the flavor-spin Fierz-identities.
In the QPC-models, used in these notes, we do not generate couplings for the axial $J^{PC} = 1^{+-}$- and tensor $J^{PC} = 2^{++}$-mesons.

The summary of the derived formulas, in the case of the $^3P_0$-model, with no QGG form-factor nor SU(6)-breaking effects, for the divers (I=1)-couplings is:

$$
\begin{aligned}
g_P/\sqrt{4\pi} &= +\pi^{-5/4}\,\gamma_{q\bar{q}}\,\frac{(m_P R_P)^{1/2}}{(\Lambda_{QPC}R_P)^2}\cdot(3\sqrt{2})\ ,\\
g_V/\sqrt{4\pi} &= +\pi^{-5/4}\,\gamma_{q\bar{q}}\,\frac{(m_V R_V)^{1/2}}{(\Lambda_{QPC}R_V)^2}\cdot(3/2\sqrt{2})\ ,\\
g_S/\sqrt{4\pi} &= +\pi^{-5/4}\,\gamma_{q\bar{q}}\,\frac{(m_S R_S)^{-1/2}}{(\Lambda_{QPC}R_S)^2}\cdot\frac{9m_S}{2M_B}\ ,\\
g_A/\sqrt{4\pi} &= -\pi^{-5/4}\,\gamma_{q\bar{q}}\,\frac{(m_A R_A)^{-1/2}}{(\Lambda_{QPC}R_A)^2}\cdot\frac{6m_A}{2M_B}.
\end{aligned}
$$

In a concise form the couplings can be written as

$$
g_{BBM}(\pm) = \frac{3}{(4\pi^3)^{1/4}}\gamma_{q\bar{q}}\ X_M(I_M,L_M)\ F^{\pm}_M(R^{\pm}_M),
$$

where $\pm = -(-)^{L_f}$ with L_f the orbital angular momentum of the final MB-state.
We discuss several corrections to these 'naive' expressions for the BBM-couplings. Notably are gluonic-corrections and hadronic-vertex corrections. The gluonic-corrections are isospin independent. This because a two-quark subsystem in a baryon is necessarily in a $\{3^*\}_c$-irrep, and higher-order gluon corrections do not change the ratio of the BBM-couplings for $I_M = 0$ and $I_M = 1$. These gluonic-corrections will produce non-harmonic effects in the quark wave-functions, similar to those responsible for the 'running' of the pair-creation constant $\gamma_{q\bar{q}}$. Therefore, the effective radii of the $q\bar{q}$-states are not constant, and may be adapted to produce rather fine-tuned descriptions of the BBM-couplings. In obtaining these descriptions we fit the weights A and B for the $^3P_0$- and $^3S_1$-mechanism. *All solutions are $^3P_0$-dominated.*

We can produce many solutions with different Λ_{QPC}-values by noting the invariance of the BBM-couplings under the scale-transformation:

$$R_M^+ = s\ R_M\ , R_M^- = s^{3/5}\ R_M\ ,\ \Lambda_{QPC} = s^{-3/4}\Lambda_{QCD},$$

where $R_M^\pm$ refer to the mesons with $g_M \sim R_M^{-3/2}$ respectively $\sim R_M^{-5/2}$.
First, in a "trial-solution", see Table II, we show a 'naive' solution with the same radius for the vector-, scalar-, and axial-vector mesons (R =0.66 fm). Notice that the I=1 BBM-couplings are too strong. Apparently, the vector- and scalar-meson I=1 couplings are enhanced over the I=0 coupling, which possibly can be attributed to the isospin-dependent gluon-corrections alluded to above.
Next, we produced two 'realistic' solutions with $^3P_0/^3S_1 = 66\%/33\%$,
$99\%/1\%$. In the first solution $s = 1$ (Table II): $\Lambda_{QPC} = 350$ MeV, $R_M^\pm(I=0) = 0.66$ fm, $R_M^\pm(I=1) = 0.86$ fm. In the second solution $s = 1/2$ (Table V): $\Lambda_{QPC} = 600$ MeV, $R_M^\pm(I=0) = 0.33$ fm, $R_M^\pm(I=1) = 0.45$ fm.
In these tables, for the spin-triplet $q\bar{q}$-states (vector- and sclaar mesons) $R_M(I=1) > R_M(I=0)$, and for the spin-singlet $q\bar{q}$-states (pseudoscalars) $R_M(I=1) < R_M(I=0)$. This might be understood as due to chiral-goldstone boson exchange between quarks , see section V F. (The axial-vector mesons seem not to follow such a rule. Also, an OBE-model for mesons gives opposite results, see section V G)
As a further possibility, we analyze in sections V C and V D *SU(6)-breaking effects*. This opens the possibilty of changing the weights of the isospin 0 and 1 component of the di-quark system in the baryons. This way the strength of the coupling to the $I_M = 0, 1$-mesons could be altered. May be this gives more natural solutions for the $I_M = 1$-problem. Exploiting ($\underline{56}$)-($\underline{70}$) SU(6)-irrep mixing, this idea is tested in section V D. The results favor the dominance of the $^3P_0$- over the $^3S_1$- mechanism with a ratio 2:1. This is in agreement with the lattice study by Isgur and Paton.
The pseudoscalar mesons are exceptional, due to the small pion mass. This means that the perturbative gluon-correction to the running pair-creation constant can not be used, and we have to modify both R_π and the running pair-creation constant γ at the pion mass in order to get the proper strength of the π-coupling.

II. INTRODUCTION

The quark-pair-creation (QPC)-model [1] gives a prediction for all BBM-couplings. This in terms of a single pair creation constant, and the radii of the quark-wave-functions of the baryons and the mesons.

We start out from the $^3S_1$ Pair-creation Hamiltonian

$$\begin{aligned}\mathcal{H}_I^{(V)} &= -\gamma_{q\bar{q}}^{(V)} \left(\sum_i \bar{q}_{i,\alpha} (\boldsymbol{\lambda})^\alpha{}_\beta \gamma^\mu q_{i,\beta}\right) \otimes \left(\sum_j \bar{q}_{j,\gamma} (\boldsymbol{\lambda})^\gamma{}_\delta \gamma_\mu q_{j,\delta}\right) \\ &= -\gamma_{q\bar{q}}^{(V)} \left(\sum_i \bar{q}_i \gamma^\mu q_i\right) \otimes \left(\sum_j \bar{q}_j \gamma_\mu q_j\right) \cdot \\ &\quad \times \left(\chi_{i,\alpha}^\dagger (\boldsymbol{\lambda})^\alpha{}_\beta \chi_{i,\beta}\right) \otimes \left(\chi_{j,\gamma}^\dagger (\boldsymbol{\lambda})^\gamma{}_\delta \chi_{j,\delta}\right) \end{aligned} \quad (1)$$

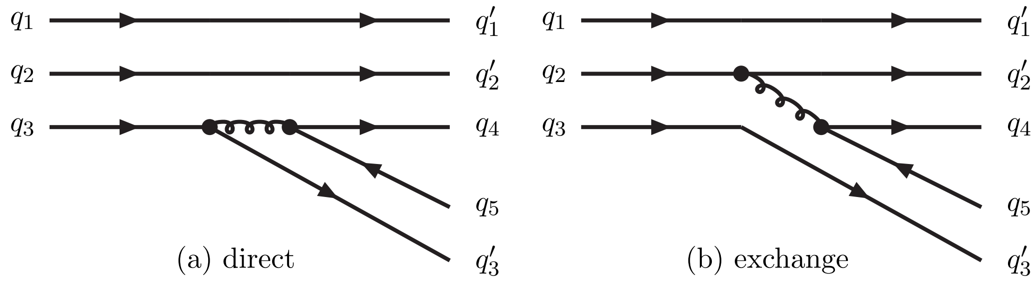

FIG. 1: $^3P_0$- *and* $^3S_1$*-quark-pair-creation (QPC)*

where $\gamma_{q\bar{q}}^{(V)}$ is a phenomenological constant, and the summations run as $i, j = u, d, s$.

NOTE 1: In this model we have in the fundamental process a (confined) scalar or gluon propagator. This implies, assuming a constant propagator, an extra factor depending on a scalar or (massive) gluon exchange

$$(-i)^2.(\mp i/m_G^2) \sim \pm i/\Lambda_{QPC}^2$$

meaning $\sim \pm i H_{int}$.

NOTE 2: The flavor part of the interaction Hamiltonian is similarly to the color part in (1), and can be treated analogously. This in particular for the Fierzing procedures. Because of the direct-product character of the spinors the flavor Fierzing is like that for the color.

Rearrangement is supposed to take place when a quark-antiquark pair is created by some mechanism in a baryon, where one quark from the baryon combines into a mesonic state with the anti-quark from the pair. The quark from the pair recombines with the two remaining quarks of the baryon to make the baryon in the final state. This rearrangements into mesons of different kind can be understood from a Fierz-transformation applied to (1). One has the identity [2]

$$\mathcal{H}_I^{(V)} = +\gamma_{q\bar{q}} \sum_{i,j} \Big[+\ \bar{q}_i\ q_j \cdot \bar{q}_i\ q_j - \frac{1}{2}\ \bar{q}_i \gamma_\mu q_j \cdot \bar{q}_j \gamma^\mu q_i$$
$$-\frac{1}{2}\ \bar{q}_i \gamma_\mu \gamma_5 q_j \cdot \bar{q}_j \gamma^\mu \gamma^5 q_i -\ \bar{q}_i \gamma_5 q_j \cdot \bar{q}_j \gamma^5 q_i \Big] . \tag{2}$$

Here, we considered only the flavor-spin Fierzing. [1] The appropriate Fierzing of the color structure is different for diagram (a) and diagram (b) in Fig. 1:

(a) For this diagram we use the identity [2]

$$^3S_1\ :\ (\boldsymbol{\lambda})^\gamma{}_\delta \cdot (\boldsymbol{\lambda})_\alpha{}^\beta = \frac{16}{9} \delta_\alpha^\gamma \delta_\delta^\beta - \frac{1}{3} (\boldsymbol{\lambda})_\alpha^\gamma \cdot (\boldsymbol{\lambda})_\delta^\beta,$$
$$^3P_1\ :\ \delta^\gamma{}_\delta \cdot \delta_\alpha{}^\beta = \frac{1}{3} \delta_\alpha^\gamma \delta_\delta^\beta + \frac{1}{2} (\boldsymbol{\lambda})_\alpha^\gamma \cdot (\boldsymbol{\lambda})_\delta^\beta. \tag{3}$$

[1] It should be noted that the terms for the couplings of the B-axial $J^{PC} = 1^{+-}$- and tensor $J^{PC} = 2^{++}$mesons are missing on the r.h.s. of (2). The same is true for the $^3P_0$-interaction (11).

Now, since the mesons are colorless, the second term in (3) may be neglected, and color gives the simple recoupling-factors (i) $^3S_1$-model: 16/9, and (ii) $^3P_0$-model: 1/3.

(b) In this diagram there is in fact a sum over q_1 and q_2. Because the baryons are colorless, we have

$$(\boldsymbol{\lambda}_1)_\alpha^{\ \beta} + (\boldsymbol{\lambda}_2)_\alpha^{\ \beta} = -(\boldsymbol{\lambda}_3)_\alpha^{\ \beta}. \tag{4}$$

Therefore, for this diagram we have, using (3), the identity

$$(\boldsymbol{\lambda}_5)^\gamma_{\ \delta} \cdot \sum_{i=1,2} (\boldsymbol{\lambda}_i)_\alpha^{\ \beta} = -\frac{16}{9}\delta^\gamma_\alpha \delta^\beta_\delta + \frac{1}{3}(\boldsymbol{\lambda}_5)^\gamma_\alpha \cdot (\boldsymbol{\lambda}_3)^\beta_\delta \tag{5}$$

Again, since the mesons are colorless, the second term in (5) may be neglected, and color gives the simple factor $-16/9$.

We find that the direct (a) and exchange (b) diagram give different color factors. Such a difference does not occur in the $^3P_0$-model. Now, it appears that the momentum overlap for type (b) is usually much smaller than for type (a), see section B for details. This can be traced back to our use of a constant propagator for the (confined) gluon. Therefore, in the following we neglect processes described in diagram (b). Then, the difference between the $^3P_0$- and $^3S_1$-model is, apart from an overall constant, exclusively given by the different coefficients in the flavor-spin Fierz-identities (2) and (10).

This form of the interaction Lagrangian suggests that we may expect that $g_\epsilon \approx -2g_\omega$, and $g_{a_0} \approx -2g_\rho$. Also, $g_\pi \approx -g_{a_0}$ and $g_{a_1} \approx -g_{a_0}/2$. In this note the details of the $^3S_1$-model are worked out. Here all questions are answered confirmatively. The techniques used are those of [3–5].

We compute the the isospin-, spin- recoupling matrix elements using the $^3P_0$ interaction Hamiltonian (11), see below, which is similar but not identical to the one implicitly used by [3–5].

In the $^3S_1$-model for the interaction Hamiltonian for the pair-creation one uses the one-gluon-exchange (OGE) model [6, 7], see Fig. 1. Considering one-gluon exchange, see Fig. 1, one derives the effective vertex [6, 7]

$$\begin{aligned}\Gamma(^3S_1) \;&=\; \pi\alpha_s(\lambda_i \cdot \lambda_j)\left\{\frac{\boldsymbol{\sigma}_j \cdot \mathbf{Q}}{2}\left(\frac{1}{m_i} + \frac{1}{m_j}\right)\right. \\ &\left. -i\frac{\boldsymbol{\sigma}_j \cdot \mathbf{k}_i}{m_i} - i\frac{\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j \cdot \mathbf{Q}}{2m_i}\right\}\; P_g(ji).\end{aligned} \tag{6}$$

Here, λ is the color index, m_ir, σ_i, and $\mathbf{k}_i$ are the mass, the spin operator, and the momentum for the quark with index i. $P_g(ji)$ is the gluon propagator between quark line i and line j. The latter we will take as a constant: $P_g(ji) \sim \delta_{ji}/m_g^2$, where the (effective) gluon mass is taken to be $m_g \approx (0.8fm^{-1}) \approx 250$ MeV [7]. We notice that the color factor for the coupling of colorless mesons to colorless baryons is always the same, and we can include this into an effective coupling γ_S, i.e.

$$\frac{\pi\alpha_s(\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j)}{m_G^2} \Rightarrow \gamma^{(V)}_{q\bar{q}}. \tag{7}$$

Here we use for the gluon a constant (confined) propagator $P_g = 1/m_G^2$. As is clear from (1) $\gamma_{q\bar{q}}$ has the dimension $[\text{MeV}]^{-2}$. Also, we notice that $m_G \approx \Lambda_{QPC}$, therefore

$$\gamma_{q\bar{q}} \longrightarrow \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}. \tag{8}$$

From graph (a) of Fig. 1 we see from momentum-conservation that

$$\mathbf{Q} = \mathbf{k}_3 - \mathbf{k}_3' = \mathbf{k}_4 + \mathbf{k}_5. \tag{9}$$

From the momentum conservation rules one now gets different dependences between the momenta as compared to the $^3P_0$-model. Hence, we have to do new overlap-integrals.

A. Comparison $^3S_1$- and $^3P_0$-QPC models

The $^3P_0$ Pair-creation Hamiltonian is

$$\mathcal{H}_I^{(S)} = -4\gamma_{q\bar{q}}^{(S)} \left(\sum_i \bar{q}_i q_i\right) \cdot \left(\sum_j \bar{q}_j q_j\right). \tag{10}$$

For this Hamiltonian the Fierz-identity reads [2]

$$\mathcal{H}_I^{(S)} = \gamma_{q\bar{q}}^{(S)} \sum_{i,j} \Big[+ \ \bar{q}_i\ q_j \cdot \bar{q}_j\ q_i + \bar{q}_i\gamma_\mu q_j \cdot \bar{q}_j\gamma^\mu q_i$$
$$-\frac{1}{2}\ \bar{q}_i\sigma_{\mu\nu}q_j \cdot \bar{q}_j\sigma^{\mu\nu}q_i - \bar{q}_i\gamma_\mu\gamma_5 q_j \cdot \bar{q}_j\gamma^\mu\gamma^5 q_i + \ \bar{q}_i\gamma_5 q_j \cdot \bar{q}_j\gamma^5 q_i\Big] \tag{11}$$

From the results for the couplings of the mesons in the $^3P_0$-model those for the $^3S_1$–model meson-couplings can be read off by comparing the coefficients in the Fierz-identities (2) and (11) for the corresponding operators. Here, we assume that the effect of color in the $^3P_0$- and $^3S_1$-model can be absorbed into $\gamma_{q\bar{q}}^{(S,V)}$, see below. For example, the prediction for the scalar-meson couplings will have the ratio $g_\epsilon(^3S_1) = \left[\gamma_{q\bar{q}}^{(V)}/\gamma_{q\bar{q}}^{(S)}\right] g_\epsilon(^3P_0)$. Apart from an overall constant, the couplings for the $^3S_1$-model can be read off from those of the $^3P_0$-model.

B. Review of the contents of these notes

In section III the BBM transition matrix elements are given for scalar, vector, pseudoscalar, and axial-vector mesons. Here, also a short note on the tensor mesons is made. In section IV the isospin factors are derived using the Fierz-identities. Section V contains results, discussion and conclusions. Here, we first discuss the vector mesons and the relation of the pair creation constant and the $\rho^0 \rightarrow e^+e^-$ annihilation constant f_ρ. Successively the following topics are addressed: (i) gluonic corrections, (ii) the γ-value for the QPC-model, (iii) comparison QPC-predictions and ESC-fit to nucleon-nucleon, (iv) isospin dependence QPC-meson-couplings. In section V C SU(6)-breaking and Di-quark structure is described. In section V D SU(6)-breaking via the (56)- and (70)-irrep mixing is described. Section V is concluded by a discussion of some miscelaneous topics as: (a) meson-radii and CS-Goldstone-boson exchange, (b) meson-radii and OBE-exchange, and (c) finally the relation of the QPC

constant γ and QCD-Sum rules.
Furthermore, these notes contain a number of appendices. In Appendix A the harmonic-oscillator momentum space wave-functions are given. In Appendix B the basic overlap integrals are derived for the BBM-transitions. This concerns the (56) → (56) baryon-transitions. In Appendix C the SU(6)-wave functions are given using Jacobian coordinates. The overlaps for (56) → (70) are derived. Appendix D contains the overlap integrals with the inclusion of gaussian baryon-baryon-gluon (BBG) form factors. These are worked out for both the (56) → (56) and (56) → (70) transitions. In Appendix E the connection and a comparison with the overlap integrals in the literature [3] is worked out. The notes are concluded by a mini review on the vector-dominance model (Appendix F), and on VPP-decay using QPC-model (Appendix G.

III. BBM TRANSITION MATRIX ELEMENTS

In this section we compute the $\langle B, M|H_{int}|A\rangle$ matrix elements for the different type of mesons. Restriction on the quark-level to process (a) in Fig. 1, using the Fierzed form of the interaction Hamiltonians in (11).

Following [8] we write the meson creation operators as

$$\begin{aligned}
J^{PC} = 0^{-+}: \quad d^\dagger_{P,I}(\mathbf{k};n) &= i \sum_{r,s=\pm} \int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ \cdot \\
&\times \widetilde{\psi}^{(L=0)}_M(\mathbf{k}_1,\mathbf{k}_2)\ \chi^{(0)}(r,s)\ b^\dagger_\alpha(\mathbf{k}_1,r)\ d^\dagger_\beta(\mathbf{k}_2,s), \qquad (12)\\
J^{PC} = 1^{--}: \ d^\dagger_{V,I}(\mathbf{k};m,n) &= \sum_{r,s=\pm} \int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ \cdot \\
&\times \widetilde{\psi}^{(L=0)}_M(\mathbf{k}_1,\mathbf{k}_2)\ \chi^{(1)}_m(r,s)\ b^\dagger_\alpha(\mathbf{k}_1,r)\ d^\dagger_\beta(\mathbf{k}_2,s), \qquad (13)\\
J^{PC} = 0^{++}: \ d^\dagger_{S,I}(\mathbf{k};m,n) &= \sum_{r,s=\pm} \int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ (-)^m \cdot \\
&\times \widetilde{\psi}^{(L=1)}_{M,m}(\mathbf{k}_1,\mathbf{k}_2)\ \chi^{(1)}_{-m}(r,s)\ b^\dagger_\alpha(\mathbf{k}_1,r)\ d^\dagger_\beta(\mathbf{k}_2,s), \qquad (14)\\
J^{PC} = 1^{++}: \ d^\dagger_{A,I}(\mathbf{k};m,n) &= \sum_{r,s=\pm} \int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ C(1,1,1;m_L,m_\sigma,m) \cdot \\
&\times \widetilde{\psi}^{(L=1)}_{M,m_L}(\mathbf{k}_1,\mathbf{k}_2)\ \chi^{(1)}_{m_\sigma}(r,s)\ b^\dagger_\alpha(\mathbf{k}_1,r)\ d^\dagger_\beta(\mathbf{k}_2,s), \qquad (15)\\
J^{PC} = 1^{+-}: \ d^\dagger_{B,I}(\mathbf{k};m,n) &= \sum_{r,s=\pm} \int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ \cdot \\
&\times \widetilde{\psi}^{(L=1)}_{M,m}(\mathbf{k}_1,\mathbf{k}_2)\ \chi^{(0)}(r,s)\ b^\dagger_\alpha(\mathbf{k}_1,r)\ d^\dagger_\beta(\mathbf{k}_2,s), \qquad (16)\\
J^{PC} = 2^{++}: \ d^\dagger_{T,I}(\mathbf{k};m,n) &= \sum_{r,s=\pm} \int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ C(1,1,2;m_L,m_\sigma,m) \cdot \\
&\times \widetilde{\psi}^{(L=1)}_{M,m_L}(\mathbf{k}_1,\mathbf{k}_2)\ \chi^{(1)}_{m_\sigma}(r,s)\ b^\dagger_\alpha(\mathbf{k}_1,r)\ d^\dagger_\beta(\mathbf{k}_2,s), \qquad (17)
\end{aligned}$$

for respectively the pseudoscalar-, vector-, scalar-, axial-vector mesons of the first (A_1 etc.) and second kind (B_1 etc.)[9], and tensor mesons.
Above, for notational reasons, we have omitted the isospin wave functions, which should be added, of course. For example, in the case of the pseudoscalars the full expression for the

meson creation operator reads actually

$$d^\dagger_{P,I}(\mathbf{k};n) \;=\; i \sum_{\alpha,\beta=\pm}\sum_{r,s=\pm} \varphi_n^{(I)}(\alpha,\beta) \;\int d^3k_1 d^3k_2\; \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\;\cdot$$
$$\times\widetilde{\psi}^{(L=0)}_M(\mathbf{k}_1,\mathbf{k}_2)\;\chi^{(0)}(r,s)\; b^\dagger_\alpha(\mathbf{k}_1,r)\; d^\dagger_\beta(\mathbf{k}_2,s), \tag{18}$$

and similarly for the other mesons.

The baryon and meson wave , harmonic oscillator, functions are

$$\begin{aligned}
\widetilde{\psi}_N(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \left(\frac{\sqrt{3}R_A^2}{\pi}\right)^{3/2} \exp\left[-\frac{R_A^2}{6}\sum_{i<j}(\mathbf{k}_i-\mathbf{k}_j)^2\right] \;, \\
\widetilde{\psi}^{(L=0)}_M(\mathbf{k}_1,\mathbf{k}_2) &= \left(\frac{R_M^2}{\pi}\right)^{3/4} \exp\left[-\frac{R_M^2}{8}(\mathbf{k}_1-\mathbf{k}_2)^2\right] \;, \\
\widetilde{\psi}^{(L=1)}_{M,m}(\mathbf{k}_1,\mathbf{k}_2) &= \frac{R_M}{\sqrt{2}}\left(\frac{R_M^2}{\pi}\right)^{3/4} \left[-\boldsymbol{\epsilon}_m\cdot(\mathbf{k}_1-\mathbf{k}_2)\right] \;.\exp\left[-\frac{R_M^2}{8}(\mathbf{k}_1-\mathbf{k}_2)^2\right] \;.
\end{aligned} \tag{19}$$

Here we used the spherical unit vectors

$$\boldsymbol{\epsilon}_{\pm 1} = \mp\frac{1}{\sqrt{2}}\left(\mathbf{e}_1 \pm i\mathbf{e}_2\right) \quad , \quad \boldsymbol{\epsilon}_0 = \mathbf{e}_3 \;. \tag{20}$$

The wave functions of the $L = 0, 1$-states in momentum space contains the spherical harmonics

$$\mathcal{Y}^0_0(\mathbf{k}) = Y^0_0(\mathbf{k}) = \sqrt{\frac{1}{4\pi}} \;,\; \mathcal{Y}^1_m(\mathbf{k}) = -\sqrt{\frac{3}{4\pi}}\;\boldsymbol{\epsilon}_m\cdot\mathbf{k} \;. \tag{21}$$

The quark annihilation and creation operators occur in the quark-field operators [11]

$$q_\alpha(x) \;=\; \sum_{r=\pm}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_q}{E(p)}}\left[b_\alpha(p,r)u(\mathbf{p},r)e^{-ip\cdot x} + d^\dagger_\alpha(p,r)v(\mathbf{p},r)e^{+ip\cdot x}\right], \tag{22}$$

where for the quark

$$u(\mathbf{p},r) = \begin{pmatrix} \cosh\frac{1}{2}\zeta\;\chi_r \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\;\sinh\frac{1}{2}\zeta\;\chi_r \end{pmatrix} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} \chi_r \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\;\chi_r \end{pmatrix}, \tag{23}$$

and for the anti-quark

$$v(\mathbf{p},r) = (-)^{1/2-r}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\;\sinh\frac{1}{2}\zeta\;\chi_{-r} \\ \cosh\frac{1}{2}\zeta\;\chi_{-r} \end{pmatrix} = (-)^{1/2-r}\sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\;\chi_{-r} \\ \chi_{-r} \end{pmatrix}. \tag{24}$$

Here we followed the phase conventions of [12] for the v-bispinor. In (17) $\chi^{(\sigma)}(r,s)$ denotes the coefficients for the right spin wave function. Explicitly, they are given for $\sigma = 0$ and $\sigma = 1$ by

$$\chi^{(0)}(r,s) \;=\; \left(\delta_{r,+1/2}\delta_{s,-1/2} - \delta_{r,-1/2}\delta_{s,+1/2}\right)/\sqrt{2}, \tag{25}$$

$$\chi^{(1)}_m(r,s) \;=\; \begin{cases} \delta_{r,+1/2}\delta_{s,+1/2} & m=1 \\ \left(\delta_{r,+1/2}\delta_{s,-1/2} + \delta_{r,-1/2}\delta_{s,+1/2}\right)/\sqrt{2} & m=0 \\ \delta_{r,-1/2}\delta_{s,-1/2} & m=-1 \end{cases} \tag{26}$$

Notice that we put the antiquark in second place, and the spin-up/down state for the antiquark has $s = \pm 1/2$.

A. Scalar mesons

First, we compute the matrix elements for the coupling of the scalar mesons.

(i) Scalar-meson-overlap matrix element:

$$\begin{aligned}\langle 0|\bar{q}_{i\gamma}(x)q_{j\delta}(x)|M_S(\mathbf{k})\rangle \;=\; &\sum_{r,s=\pm}\int d^3k_1 d^3k_2\;\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\;(-)^m\;\widetilde{\psi}^{(L=1)}_{M,m}(\mathbf{k}_1,\mathbf{k}_2)\,\cdot \\ &\times\chi^{(1)}_{-m}(r,s)\;\langle 0|\bar{q}_{i\gamma}(x)q_{j\delta}(x)\;b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle.\end{aligned} \tag{27}$$

The vacuum expectation is given as

$$\begin{aligned}&\langle 0|\bar{q}_{i\gamma}(\mathbf{x})q_{j\delta}(\mathbf{x})\;b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle = \delta_{\gamma\beta}\delta_{\delta\alpha}(2\pi)^{-3}\sqrt{\frac{m_i}{E_i(k_1)}\frac{m_j}{E_j(k_2)}}\,\cdot \\ &\times\left[\bar{v}(\mathbf{k}_2,s)\;u(\mathbf{k}_1,r)\right]\;\exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right].\end{aligned} \tag{28}$$

The spinor matrix element gives

$$\begin{aligned}\left[\bar{v}(\mathbf{k}_2,s)\;u(\mathbf{k}_1,r)\right] \;&=\; (-)^{1/2-s}\sqrt{\frac{(E_i(\mathbf{k}_1)+m_i)}{2m_i}\frac{(E_j(\mathbf{k}_2)+m_j)}{2m_j}}\,\cdot \\ &\quad\times\chi^\dagger_{-s}\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_2}{E_j(\mathbf{k}_2)+m_j} - \frac{\boldsymbol{\sigma}\cdot\mathbf{k}_1}{E_i(\mathbf{k}_1)+m_i}\right]\chi_r \\ &\approx\; \frac{(-)^{1/2-s}}{2m_Q}\chi^\dagger_{-s}\left[\boldsymbol{\sigma}\cdot(\mathbf{k}_2-\mathbf{k}_1)\right]\chi_r,\end{aligned} \tag{29}$$

where we used the non-relativistic approximation, using constituent quarks, and m denotes the mean mass of the quarks. Next, we note that

$$\begin{aligned}\chi^\dagger_j(+1/2)\;\boldsymbol{\sigma}\;\chi(+1/2) \;&=\; +\boldsymbol{\epsilon}_z \\ \chi^\dagger_j(+1/2)\;\boldsymbol{\sigma}\;\chi(-1/2) \;&=\; \boldsymbol{\epsilon}_x - i\boldsymbol{\epsilon}_y \\ \chi^\dagger_j(-1/2)\;\boldsymbol{\sigma}\;\chi(+1/2) \;&=\; \boldsymbol{\epsilon}_x + i\boldsymbol{\epsilon}_y \\ \chi^\dagger_j(-1/2)\;\boldsymbol{\sigma}\;\chi(-1/2) \;&=\; \text{-}\boldsymbol{\epsilon}_z.\end{aligned}$$

From this we derive that

$$\sum_{r,s=\pm} (-)^{1/2-s} \chi_m^{(1)}(r,s)\ \left[\chi_j^\dagger(-s)\ \boldsymbol{\sigma}\ \chi_i(r)\right] = \begin{cases} m = +1\ :\ +(\boldsymbol{\epsilon}_x + i\boldsymbol{\epsilon}_y) & = -\sqrt{2}\ \boldsymbol{\epsilon}_{+1} \\ m = \ \ 0\ :\ \quad -\sqrt{2}\boldsymbol{\epsilon}_z & = -\sqrt{2}\ \boldsymbol{\epsilon}_0 \\ m = -1\ :\ -(\boldsymbol{\epsilon}_x - i\boldsymbol{\epsilon}_y) & = -\sqrt{2}\ \boldsymbol{\epsilon}_{-1} \end{cases} \tag{30}$$

which gives

$$(-)^m \Delta_{M,-m} \sum_{r,s=\pm} (-)^{1/2-s} \chi_m^{(1)}(r,s)\ \left[\chi_j^\dagger(-s)\ \boldsymbol{\sigma}\cdot\boldsymbol{\Delta}_M\ \chi_i(r)\right] = -\sqrt{2}\ \boldsymbol{\Delta}_M^2,$$

where $\boldsymbol{\Delta}_M = \mathbf{k}_1 - \mathbf{k}_2$.

Using these results, we get for the meson-overlap matrix element (27)

$$\langle 0|\bar{q}_i(\mathbf{x})q_j(\mathbf{x})|M_S(\mathbf{k})\rangle = \frac{R_M}{2m_q}\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ (\mathbf{k}_1-\mathbf{k}_2)^2 \cdot \\ \times(2\pi)^{-3}\ \widetilde{\psi}_{M,m}^{(L=0)}(\mathbf{k}_1,\mathbf{k}_2)\ \exp\left[+i\,(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}\right]. \tag{31}$$

(ii) Baryon-overlap matrix element: The quark-transition matrix element for the scalar operator is

$$\langle q_\beta(\mathbf{k}_4)|\bar{q}_{j\delta}(\mathbf{x})q_{i\gamma}(\mathbf{x})|q_\alpha(\mathbf{q}_3)\rangle = (2\pi)^{-3}\sqrt{\frac{(E_j(\mathbf{k}_4)+m_j)}{2m_j}\frac{(E_i(\mathbf{k}_3)+m_i)}{2m_i}}\cdot \\ \times\chi_j\dagger\left\{1-\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_4}{E_j(\mathbf{k}_4)+m_j}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_3}{E_i(\mathbf{k}_3)+m_i}\right\}\chi_i\ \exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right] \\ \approx (2\pi)^{-3}\delta_{\gamma\alpha}\delta_{\delta\beta}\ \exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right], \tag{32}$$

where again we used the non-relativistic approximation. Folding this matrix element, we get

$$\langle B(\mathbf{p}')|\bar{q}_j(\mathbf{x})q_i(\mathbf{x})|B(\mathbf{p})\rangle = \delta_{ij}(2\pi)^{-3}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot \\ \times\int d^3k_3' d^3k_4\ \delta(\mathbf{p'}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot\widetilde{\psi}_B^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\cdot \\ \times\exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right]. \tag{33}$$

Combining factors we get finally for the transition matrix

$$\langle B(\mathbf{p}'),M_S(\mathbf{k})|H_{int}|B(\mathbf{p})\rangle = \int d^3x\ \langle B(\mathbf{p}'),M_S(\mathbf{k})|\mathcal{H}_{int}(\mathbf{x})|B(\mathbf{p})\rangle \\ \Rightarrow (2\pi)^{-3}\frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}(\frac{R_M}{\sqrt{2}})\frac{3}{\sqrt{2}M_B}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot \\ \times\int d^3k_3' d^3k_4 d^3k_5\delta(\mathbf{k_3}-\mathbf{k_3'}-\mathbf{k_4}-\mathbf{k_5})\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot \\ \times\widetilde{\psi}_B^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_3',\mathbf{k}_5)^*\ (\mathbf{k}_3'-\mathbf{k}_5)^2, \tag{34}$$

where we defined $M_B \equiv 3m_q$. From Appendix B it is easily seen that the overlap-integral in (34) is given by

$$-\frac{\partial}{\partial\gamma} I_0^{dir}(A;B,M) \stackrel{\mathbf{k}=0}{\longrightarrow} 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} \cdot \frac{12}{R_M^2}.$$

Note that in the execution of the γ-differentiation γ and R_M have to be treated as independent variables. Then, the scalar BBM-coupling for the $^3P_0$-model is given by the matrix element

$$\langle B(\mathbf{0}'), M_S(\mathbf{0})|H_{int}|B(\mathbf{0})\rangle = \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{3}{\sqrt{2}M_B}(\frac{R_M}{\sqrt{2}}) \cdot (2\pi)^{-3} \cdot$$
$$\times \frac{12}{R_M^2} \cdot 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} = (2\pi)^{-3} \cdot \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{72}{2\sqrt{2}M_B R_M}\left(\frac{\pi}{R_M^2}\right)^{3/4}. \tag{35}$$

The hadron level interaction Hamiltonian density for the $a_0(980)$-meson to the baryons is

$$\mathcal{H}_I = g_{NNa_0}[\bar{\psi}(x)\boldsymbol{\tau}\psi(x)] \cdot \mathbf{a}_0(x). \tag{36}$$

giving the matrix element, apart from the isospin-factor,

$$\langle a_0(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\omega(\mathbf{k})}}\sqrt{\frac{M^2}{E_{p'}E_p}} \cdot g_{NNa_0} \cdot$$
$$\times \left[\bar{u}(\mathbf{p}')u(\mathbf{p})\right]\ \delta(\mathbf{p}-\mathbf{p}'-\mathbf{k})\ . \tag{37}$$

Using the non-relativistic approximation (N.R.) (37) leads to

$$\langle a_0(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle\ =\ \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2m_\epsilon}} \cdot g_{NNa_0}\ \left(\chi^\dagger_{s'_z}\ \chi_{s_z}\right) \cdot$$
$$\times\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k}). \tag{38}$$

From (38) etc. one obtains upon the comparison of (38) with (35) for the NNa_0-coupling, including the isospin factor $1/\sqrt{2}$, the expression

$$g_S\ =\ (2\pi)^{-3} \cdot \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{72}{2\sqrt{2}M_B R_M}\left(\frac{\pi}{R_M^2}\right)^{3/4}(2\pi)^{3/2}\sqrt{2m_S}/\sqrt{2}$$
$$=\ \gamma_{q\bar{q}}\,\pi^{-3/4}\frac{\sqrt{m_S R_M}}{(\Lambda_{QPC}R_M)^2}\frac{9}{M_B R_M} \sim R_M^{-5/2}, \tag{39}$$

where we denoted $g_{Nna_0} = g_S$. Then, for $\Lambda_{QPC} = 250$ MeV, $R_M = 0.67$ fm, $m_S = m_{a_0} = 962$ MeV, $M_B = 940$ MeV we get for the rationalized coupling $g_S/\sqrt{4\pi} = 0.85\gamma_{q\bar{q}} \approx 1.31$, Here, in the last step we used $\gamma_{q\bar{q}} \approx 1.53$. which is a very realistic value! Note that $g_{\epsilon NN}/\sqrt{4\pi} = 3g_S/\sqrt{4\pi} \approx 3.63$.

B. Vector-mesons I

We compute the ω-meson direct-coupling g_ω, and take the $\mu = 0$-component in the vector type operators in the interaction Hamiltonians (1) and (11). (For the $^3S_1$-state one has to use the $\boldsymbol{\rho}$-vector, see next subsection.)

> *NOTE: (i) From the relation $k \cdot \rho = k_0\rho_0 - \mathbf{k} \cdot \boldsymbol{\rho} = 0$, where $\rho^\mu(\lambda)$ is the ω polarization vector, one infers that under the parity transformation ρ_0 is scalar-like. Then, in first instance one would be inclined to treat ρ_0 as a $^3P_0$ $q\bar{q}$-state! Consequently, the NR-treatment seems similar to the scalar-meson case. (ii) However the scalar and vector states behave differently under charge-conjugation. From $C\boldsymbol{\rho}(\mathbf{k}) = -\boldsymbol{\rho}(\mathbf{k})$, it follows that also $C\rho_0(\mathbf{k}) = -\rho_0(\mathbf{k})$, and therefore can not be represented by a $^3P_0$-state.*

Below, we will show that the assumption of a $^3P_0$-state for ρ_0 leads to a conflict indeed.

(i) Vector-meson-overlap matrix element for $\mu = 0$:

$$\langle 0|q_i^\dagger(x)q_j(x)|M_V(\mathbf{k},\epsilon_0)\rangle = \sum_{r,s=\pm}\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ (-)^m \widetilde{\psi}^{(L=1)}_{M,m}(\mathbf{k}_1,\mathbf{k}_2)\cdot \times\chi^{(1)}_{-m}(r,s)\ \langle 0|q_i^\dagger(x)q_j(x)\ b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle \tag{40}$$

The vacuum expectation is given as

$$\langle 0|q_i^\dagger(\mathbf{x})q_j(\mathbf{x})\ b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle = \delta_{i\beta}\delta_{j\alpha}(2\pi)^{-3}\sqrt{\frac{m_i}{E_i(k_1)}\frac{m_j}{E_j(k_2)}}\cdot \times\left[v^\dagger(\mathbf{k}_2,s)\ u(\mathbf{k}_1,r)\right]\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right]. \tag{41}$$

The spinor matrix element gives

$$\left[v^\dagger(\mathbf{k}_2,s)\ u(\mathbf{k}_1,r)\right] = (-)^{1/2-s}\sqrt{\frac{(E_i(\mathbf{k}_1)+m_i)}{2m_i}\frac{(E_j(\mathbf{k}_2)+m_j)}{2m_j}}\cdot \times\chi^\dagger_{-s}\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_2}{E_j(\mathbf{k}_2)+m_j}+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_1}{E_i(\mathbf{k}_1)+m_i}\right]\chi_r \approx \frac{(-)^{1/2-s}}{2m_q}\chi^\dagger_{-s}\left[\boldsymbol{\sigma}\cdot(\mathbf{k}_2+\mathbf{k}_1)\right]\chi_r = \frac{(-)^{1/2-s}}{2m_q}\chi^\dagger_{-s}\left[\boldsymbol{\sigma}\cdot\mathbf{k}_M\right]\chi_r, \tag{42}$$

where we used the non-relativistic approximation, using constituent quarks, and m denotes the mean mass of the quarks. Also, we denoted the meson related momenta by $\mathbf{k}_M \equiv \mathbf{k}_1+\mathbf{k}_2$ and $\boldsymbol{\Delta}_M \equiv \mathbf{k}_1-\mathbf{k}_2$. Next, we use (30) which gives

$$(-)^m\Delta_{M,-m}\sum_{r,s=\pm}(-)^{1/2-s}\chi^{(1)}_m(r,s)\ \left[\chi^\dagger(-s)\ \boldsymbol{\sigma}\cdot\mathbf{k}_M\ \chi(r)\right] = -\sqrt{2}\boldsymbol{\Delta}_M\cdot\mathbf{k}_M. \tag{43}$$

(ii) Baryon-overlap matrix element: The quark-transition matrix element for the vector operator is

$$\langle q_\beta(\mathbf{k}_4)|q_j^\dagger(\mathbf{x})q_i(\mathbf{x})|q_\alpha(\mathbf{q}_3)\rangle = (2\pi)^{-3}\sqrt{\frac{(E_j(\mathbf{k}_4)+m_j)}{2m_j}\frac{(E_i(\mathbf{k}_3)+m_i)}{2m_i}}\cdot$$
$$\times\chi_j\dagger\left\{1+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_4}{E_j(\mathbf{k}_4)+m_j}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_3}{E_i(\mathbf{k}_3)+m_i}\right\}\chi_i\ \exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big]$$
$$\approx(2\pi)^{-3}\delta_{i\beta}\delta_{j\alpha}\ \exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big], \tag{44}$$

where again we used the non-relativistic approximation, which is equal to that for scalar mesons cfrm. (32). Folding this matrix element, we get

$$\langle B(\mathbf{p}')|q_j^\dagger(\mathbf{x})q_i(\mathbf{x})|B(\mathbf{p})\rangle = \delta_{i\alpha}\delta_{j\beta}(2\pi)^{-3}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot$$
$$\times\int d^3k_3'd^3k_4\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot\widetilde{\psi}_B^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\cdot$$
$$\times\exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big]. \tag{45}$$

Combining factors we get finally for the transition matrix

$$\langle B(\mathbf{p}'),M_V(\mathbf{k})|H_{int}|B(\mathbf{p})\rangle = \int d^3x\ \langle B(\mathbf{p}'),M_V(\mathbf{k})|\mathcal{H}_{int}(\mathbf{x})|B(\mathbf{p})\rangle$$
$$\Rightarrow(2\pi)^{-3}\ \frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}}\sqrt{2}R_M\ \frac{3}{\sqrt{2}M_B}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot$$
$$\times\int d^3k_3'd^3k_4d^3k_5\delta(\mathbf{k_3}-\mathbf{k_3'}-\mathbf{k_4}-\mathbf{k_5})\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot$$
$$\times\widetilde{\psi}_B^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_3',\mathbf{k}_5)^*\ \left(\mathbf{k}_3'^2-\mathbf{k}_5^2\right). \tag{46}$$

We note that $(\mathbf{k}_3'^2-\mathbf{k}_5^2)=\mathbf{k}_M\cdot\boldsymbol{\Delta}_M$. **However, this means that, see Appendix B, the overlap-integral (46) gives zero!? This illustrates the impossibility of representing ρ_0 by a $^3P_0$-state.**

C. Vector mesons II

We compute the ω-meson direct-coupling g_ω, and take the $\mu=i$-components in the vector type operators in the interaction Hamiltonians (1) and (11).

(i) Vector-meson-overlap matrix element for $\mu=i$:

$$\langle 0|\bar{q}_i(x)\boldsymbol{\gamma}\ q_j(x)|M_V(\mathbf{k},\boldsymbol{\epsilon}_m)\rangle\ =\ \sum_{r,s=\pm}\int d^3k_1d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_1,\mathbf{k}_2)\cdot$$
$$\times\chi_m^{(1)}(r,s)\ \langle 0|q_i^\dagger(x)q_j(x)\ b^\dagger(\mathbf{k}_1,r)d^\dagger(\mathbf{k}_2,s)|0\rangle \tag{47}$$

The vacuum expectation is given as

$$\langle 0|\bar{q}_i(\mathbf{x})\boldsymbol{\gamma}\ q_j(\mathbf{x})\ b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle = \delta_{i\beta}\delta_{j\alpha}(2\pi)^{-3}\sqrt{\frac{m_i}{E_i(k_1)}\frac{m_j}{E_j(k_2)}}\cdot$$
$$\times\left[\bar{v}(\mathbf{k}_2,s)\ \boldsymbol{\gamma}\ u(\mathbf{k}_1,r)\right]\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right]. \tag{48}$$

The spinor matrix element gives

$$\begin{aligned}\left[\bar{v}(\mathbf{k}_2,s)\ \boldsymbol{\gamma}\ u(\mathbf{k}_1,r)\right] &= (-)^{1/2-s}\sqrt{\frac{(E_i(\mathbf{k}_1)+m_i)}{2m_i}\frac{(E_j(\mathbf{k}_2)+m_j)}{2m_j}}\cdot\\ &\quad\times\chi^\dagger_{-s}\left[\boldsymbol{\sigma}+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_2\boldsymbol{\sigma}\boldsymbol{\sigma}\cdot\mathbf{k}_1}{(E_j(\mathbf{k}_2)+m_j)(E_i(\mathbf{k}_1)+m_i)}\right]\chi_r\\ &\approx (-)^{1/2-s}\left[\chi^\dagger_{-s}\ \boldsymbol{\sigma}\ \chi_r\right],\end{aligned} \tag{49}$$

where again we used the non-relativistic approximation. Also, we denoted the meson momentum by $\mathbf{k}_M \equiv \mathbf{k}_1 - \mathbf{k}_2$. Next, we use (30) which gives

$$\sum_{r,s=\pm}(-)^{1/2-s}\chi^{(1)}_m(r,s)\ \left[\chi^\dagger(-s)\ \boldsymbol{\sigma}\ \chi(r)\right] = -\sqrt{2}\ \boldsymbol{\epsilon}_m. \tag{50}$$

(ii) Baryon-overlap matrix element: The quark-transition matrix element for the vector operator is

$$\begin{aligned}&\langle q_\beta(\mathbf{k}_4)|\bar{q}_j(\mathbf{x})\boldsymbol{\gamma}\ q_i(\mathbf{x})|q_\alpha(\mathbf{q}_3)\rangle = (2\pi)^{-3}\sqrt{\frac{(E_j(\mathbf{k}_4)+m_j)}{2m_j}\frac{(E_i(\mathbf{k}_3)+m_i)}{2m_i}}\cdot\\ &\times\chi_j\dagger\left\{\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_4\boldsymbol{\sigma}}{E_j(\mathbf{k}_4)+m_j}+\frac{\boldsymbol{\sigma}\boldsymbol{\sigma}\cdot\mathbf{k}_3}{E_i(\mathbf{k}_3)+m_i}\right\}\chi_i\ \exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right]\\ &\approx (2\pi)^{-3}\frac{\delta_{i\alpha}\delta j\beta}{2m_q}\ \chi^\dagger_{s'}\left[(\mathbf{k}_3+\mathbf{k}_4)+i\mathbf{k}_M\times\boldsymbol{\sigma}\right]\chi_s\ \exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right],\end{aligned} \tag{51}$$

where again we used the non-relativistic approximation, which is equal to that for scalar mesons cfrm. (32). Folding this matrix element, we get

$$\begin{aligned}&\langle B(\mathbf{p}')|\bar{q}_j(\mathbf{x})\ \boldsymbol{\gamma}\ q_i(\mathbf{x})|B(\mathbf{p})\rangle = \delta_{ij}(2\pi)^{-3}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot\\ &\times\int d^3k'_3d^3k_4\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot\widetilde{\psi}^*_B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\cdot\\ &\times\exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right]\cdot\frac{1}{2m_q}\ \chi^\dagger_{s'}\left[(\mathbf{k}_3+\mathbf{k}_4)+i\mathbf{k}_M\times\boldsymbol{\sigma}\right]\chi_s.\end{aligned} \tag{52}$$

Combining factors we get finally for the transition matrix

$$\langle B(\mathbf{p}'),M_V(\mathbf{k},m)|H_{int}|B(\mathbf{p})\rangle = \int d^3x\ \langle B(\mathbf{p}'),M_V(\mathbf{k})|\mathcal{H}_{int}(\mathbf{x})|B(\mathbf{p})\rangle$$

$$\Rightarrow (2\pi)^{-3}\ \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\sqrt{2}\ \frac{3}{2M_B}\chi_{s_f}^\dagger \left[2\mathbf{q} + i\mathbf{k}\times\boldsymbol{\sigma}\right]\chi_{s_i}\cdot\cdot$$
$$\times \int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot$$
$$\times \int d^3k_3' d^3k_4 d^3k_5 \delta(\mathbf{k_3}-\mathbf{k_3'}-\mathbf{k_4}-\mathbf{k_5})\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot$$
$$\times\widetilde{\psi}_B^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_3',\mathbf{k}_5)^*. \tag{53}$$

From Appendix B it is easily seen that the overlap-integral in (53) is given by

$$I_0^{dir}(A;B,M) \stackrel{\mathbf{k}=0}{\longrightarrow} \frac{1}{8}\left(\frac{2\pi}{R_A^2}\right)^{3/2}\left(\frac{4\pi^2}{3R_A^2R_M^2}\right)^{3/2}\mathcal{N}_0 = 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4}.$$

The direct vector BBM-coupling is given by the matrix element

$$\langle B(-\mathbf{k}), M_{V,0}(\mathbf{k})|H_{int}|B(\mathbf{0})\rangle \Rightarrow \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{3}{2M_B}\sqrt{2}\cdot(2\pi)^{-3}\cdot 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4}.$$
$$\Rightarrow (2\pi)^{-3}\ \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{12}{2M_B}\left(\frac{\pi}{R_M^2}\right)^{3/4}\cdot\chi_{s_f}^\dagger\left[\boldsymbol{\epsilon}\cdot\mathbf{k} + i\boldsymbol{\epsilon}\times\mathbf{k}\cdot\boldsymbol{\sigma}\right]\chi_{s_i}, \tag{54}$$

where $\mathbf{p}=0, \mathbf{p}'=-\mathbf{k}$, and $\mathbf{q}=\mathbf{k}/2$.
The hadron level interaction Hamiltonian density for the ω-meson to the baryons is

$$\mathcal{H}_I = g_V[\bar{\psi}(x)\gamma_\mu\psi(x)]\ \omega^\mu(x). \tag{55}$$

giving the matrix element

$$\langle\omega^\mu(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\omega(\mathbf{k})}}\sqrt{\frac{M^2}{E_{p'}E_p}}\cdot g_V\cdot$$
$$\times\left[\bar{u}(\mathbf{p}')\gamma_\mu u(\mathbf{p})\cdot\epsilon^\mu(\mathbf{k}_M)\right]\ \delta(\mathbf{p}-\mathbf{p}'-\mathbf{k})\ . \tag{56}$$

Using the non-relativistic approximation (N.R.) (48) leads to

$$\langle\boldsymbol{\omega}(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle\ =\ \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2m_V}}\cdot g_V\ \chi_{s_z'}^\dagger\ \left\{+\frac{\boldsymbol{\epsilon}\cdot\mathbf{q}}{M}+\frac{i}{2M}\boldsymbol{\epsilon}\times\mathbf{k}\cdot\boldsymbol{\sigma}\right\}\chi_{s_z}\cdot$$
$$\times\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k}). \tag{57}$$

From (57) etc. one obtains upon the comparison of (57) with (54) for the $NN\omega$-coupling, including the isospin factor $1/\sqrt{2}$, the expression

$$g_V\ =\ (2\pi)^{-3}\cdot\frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{12M_B}{2M_B}\left(\frac{\pi}{R_M^2}\right)^{3/4}(2\pi)^{3/2}\sqrt{2m_V}/\sqrt{2}$$
$$=\ (3/\sqrt{2})\ \pi^{-3/4}\gamma_{q\bar{q}}\ \frac{\sqrt{m_VR_M}}{(\Lambda_{QPC}R_M)^2}\sim R_M^{-3/2}. \tag{58}$$

Comparing this with (39) we find for the ratio

$$g_V/g_S = \frac{1}{3\sqrt{2}}\ \sqrt{\frac{m_V}{m_S}}\ (M_BR_M). \tag{59}$$

For $m_V = m_S = 750$ MeV, and $R_M = 0.7$ fm, we have $g_V/g_S = 0.85$, which is conform the expectation based on the Fierz-identities in the $^3P_0$-model.

D. Pseudoscalar mesons

We compute the pseudo-scalar-meson, and take the γ_5-type operators in the interaction Hamiltonians (1) and (11).

(i) Pseudoscalar-meson-overlap matrix element:

$$\begin{aligned}\langle 0|\bar{q}_i(x)\gamma_5 q_j(x)|M_P(\mathbf{k})\rangle &= i\sum_{r,s=\pm}\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_1,\mathbf{k}_2)\cdot \\ &\times\chi^{(0)}(r,s)\ \langle 0|\bar{q}_i(x)\gamma_5 q_j(x)\ b^\dagger(\mathbf{k}_1,r)d^\dagger(\mathbf{k}_2,s)|0\rangle\end{aligned} \tag{60}$$

The vacuum expectation is given as

$$\begin{aligned}&\langle 0|\bar{q}_i(\mathbf{x})\gamma_5\ q_j(\mathbf{x})\ b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle = \delta_{i\beta}\delta j\alpha(2\pi)^{-3}\sqrt{\frac{m_i}{E_i(k_1)}\frac{m_j}{E_j(k_2)}}\cdot \\ &\times\left[\bar{v}(\mathbf{k}_2,s)\ \gamma_5\ u(\mathbf{k}_1,r)\right]\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right].\end{aligned} \tag{61}$$

The spinor matrix element gives

$$\begin{aligned}\left[\bar{v}(\mathbf{k}_2,s)\ \gamma_5 u(\mathbf{k}_1,r)\right] &= (-)^{1/2-s}\sqrt{\frac{(E_i(\mathbf{k}_1)+m_i)}{2m_i}\frac{(E_j(\mathbf{k}_2)+m_j)}{2m_j}}\cdot \\ &\times\chi^\dagger_{-s}\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_2}{E_j(\mathbf{k}_2)+m_j}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_1}{E_i(\mathbf{k}_1)+m_i}-1\right]\chi_r \\ &\approx -(-)^{1/2-s}\chi^\dagger_{-s}\chi_r = -(-)^{1/2-s}\delta_{r,-s},\end{aligned} \tag{62}$$

where we used again the non-relativistic approximation. Then, from (25) it follows that

$$\sum_{r,s=\pm}\chi^{(0)}(r,s)\ \left[\bar{v}(\mathbf{k}_2,s)\ \gamma_5 u(\mathbf{k}_1,r)\right] = +\sqrt{2}. \tag{63}$$

(ii) Baryon-overlap matrix element: The quark-transition matrix element for the pseudoscalar operator is

$$\begin{aligned}&\langle q_\beta(\mathbf{k}_4)|\bar{q}_j(\mathbf{x})\gamma_5\ q_i(\mathbf{x})|q_\alpha(\mathbf{q}_3)\rangle = (2\pi)^{-3}\sqrt{\frac{E_j(\mathbf{k}_4)+m_j}{2m_j}\frac{E_i(\mathbf{k}_3)+m_i}{2m_i}}\cdot \\ &\times\chi^\dagger_j\left\{\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_4}{E_j(\mathbf{k}_4)+m_j}-\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_3}{E_i(\mathbf{k}_3)+m_i}\right\}\chi_i\ \exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big] \\ &\approx (2\pi)^{-3}\frac{\delta_{i\alpha}\delta_{j\beta}}{2m_q}\ \chi^\dagger_{s_4}\Big[\boldsymbol{\sigma}_3\cdot(\mathbf{k}_3-\mathbf{k}_4)\Big]\chi_{s_3}\ \exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big],\end{aligned} \tag{64}$$

again using the non-relativistic approximation. Folding this matrix element, we get

$$\begin{aligned}&\langle B(\mathbf{p}')|\bar{q}_j(\mathbf{x})\gamma_5 q_i(\mathbf{x})|B(\mathbf{p})\rangle = \frac{\delta_{ij}}{2m_q}(2\pi)^{-3}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot \\ &\times\int d^3k_3' d^3k_4\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot\widetilde{\psi}^*_B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\cdot \\ &\times\exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big]\cdot\chi^\dagger_{s_j}\Big[\boldsymbol{\sigma}_i\cdot\mathbf{k}\Big]\chi_{s_i},\end{aligned} \tag{65}$$

where we used that $\mathbf{k}_3 - \mathbf{k}_4 = \mathbf{p} - \mathbf{p}' = \mathbf{k}$. Combining factors, and summing over the three quarks of the baryon $\sum_{i=1,3}$, we get finally for the transition matrix

$$
\begin{aligned}
&\langle B(\mathbf{p}'), M_P(\mathbf{k})|H_{int}|B(\mathbf{p})\rangle = \int d^3x\ \langle B(\mathbf{p}'), M_P(\mathbf{k})|\mathcal{H}_{int}(\mathbf{x})|B(\mathbf{p})\rangle \\
&\Rightarrow i\ (2\pi)^{-3}\frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}}\ \frac{3\sqrt{2}}{2M_B}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot \\
&\times\int d^3k'_3 d^3k_4 d^3k_5 \delta(\mathbf{k_3}-\mathbf{k'_3}-\mathbf{k_4}-\mathbf{k_5})\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot \\
&\times\widetilde{\psi}^*_B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}'_3,\mathbf{k}_5)^*.\chi^\dagger_{s_f}\left[\boldsymbol{\sigma}\cdot\mathbf{k}\right]\chi_{s_i},
\end{aligned} \tag{66}
$$

From Appendix B it is easily seen that the overlap-integral in (66) is given by

$$
I_0^{dir}(A;B,M) \xrightarrow{\mathbf{k}=0} \frac{1}{8}\left(\frac{2\pi}{R_A^2}\right)^{3/2}\left(\frac{4\pi^2}{3R_A^2R_M^2}\right)^{3/2}\mathcal{N}_0 = 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4}.
$$

The pseudoscalar BBM-coupling is, apart from the spinor factor $\chi^\dagger_{s_f}\left[\boldsymbol{\sigma}\cdot\mathbf{k}\right]\chi_{s_i}$, given by the matrix element

$$
\begin{aligned}
&\langle B(\mathbf{0}'), M_{P,0}(\mathbf{0})|H_{int}|B(\mathbf{0})\rangle \sim +i\frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}}\frac{3\sqrt{2}}{2M_B}\cdot(2\pi)^{-3}\cdot \\
&\times\ 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} = +i(2\pi)^{-3}\frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}}\frac{12}{2M_B}\left(\frac{\pi}{R_M^2}\right)^{3/4}.
\end{aligned} \tag{67}
$$

The hadron level interaction Hamiltonian density for the η-meson to the baryons is

$$
\mathcal{H}_I = ig_P[\bar{\psi}(x)\gamma_5\psi(x)]\ \phi(x). \tag{68}
$$

giving the matrix element

$$
\begin{aligned}
&\langle\eta(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\omega(\mathbf{k})}}\sqrt{\frac{M^2}{E_{p'}E_p}}\cdot ig_P\cdot \\
&\times\left[\bar{u}(\mathbf{p}')\gamma_5 u(\mathbf{p})\right]\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k})\ .
\end{aligned} \tag{69}
$$

Using the non-relativistic approximation (N.R.) (69) leads to

$$
\begin{aligned}
\langle\eta(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle\ &=\ \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2m_P}}\cdot\frac{ig_P}{2M_B}\ \left(\chi^\dagger_{s_f}\left[\boldsymbol{\sigma}\cdot\mathbf{k}\right]\chi_{s_i}\right) \\
&\quad\times\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k}).
\end{aligned} \tag{70}
$$

From (70) etc. one obtains upon the comparison of (70) with (67) for the $NN\eta$-coupling including the isospin factor $1/\sqrt{2}$, the expression

$$
\begin{aligned}
g_P\ &=\ (2\pi)^{-3}\cdot\frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}}\cdot 12\cdot\left(\frac{\pi}{R_M^2}\right)^{3/4}(2\pi)^{3/2}\sqrt{2m_P}/\sqrt{2} \\
&=\ 6\sqrt{2}\ \pi^{-3/4}\gamma_{q\bar{q}}\ \frac{\sqrt{m_P R_M}}{(\Lambda_{QPC}R_M)^2}\sim R_M^{-3/2}.
\end{aligned} \tag{71}
$$

Comparing this with (39) we find for the ratio

$$g_P/g_S = \frac{1}{6}\sqrt{2}\,\sqrt{\frac{m_P}{m_S}}\ (M_B R_M). \tag{72}$$

For $m_P = 140$ MeV, $m_S = 962$ MeV, and $R_M = 0.7$ fm, we have $g_P/g_S = 0.3$, which is less than the (naive) expectation based on the Fierz-identities in the $^3P_0$-model. Here the smallness of the pion mass plays its role.

E. Axial-vector mesons I

First, we compute the matrix elements for the coupling of the axial-vecror mesons.

(i) Axial-meson-overlap matrix element:

$$\langle 0|\bar{q}_i(x)\gamma_\mu\gamma_5\ q_j(x)|M_A(\mathbf{k},m)\rangle = C(1,1,1;m_L,m_\sigma,m)\sum_{r,s=\pm}\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\cdot$$
$$\times\widetilde{\psi}^{(L=1)}_{M,m_L}(\mathbf{k}_1,\mathbf{k}_2)\cdot\chi^{(1)}_{m_\sigma}(r,s)\ \langle 0|\bar{q}_i(x)\gamma^\mu\gamma_5\ q_j(x)\ b^\dagger(\mathbf{k}_1,r)d^\dagger(\mathbf{k}_2,s)|0\rangle. \tag{73}$$

The vacuum expectation is given as

$$\langle 0|\bar{q}_i(\mathbf{x})\gamma^\mu\gamma_5\ q_j(\mathbf{x})\ b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle = \delta_{i\beta}\delta_{j\alpha}(2\pi)^{-3}\sqrt{\frac{m_i}{E_i(k_1)}\frac{m_j}{E_j(k_2)}}\cdot$$
$$\times\left[\bar{v}(\mathbf{k}_2,s)\ \gamma^\mu\gamma_5\ u(\mathbf{k}_1,r)\right]\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right]. \tag{74}$$

The Dirac bispinor matrix element gives, for the space components,

$$\begin{aligned}
\left[\bar{v}(\mathbf{k}_2,s)\ \boldsymbol{\gamma}\gamma_5\ u(\mathbf{k}_1,r)\right] &= (-)^{1/2-s}\sqrt{\frac{E_i(\mathbf{k}_1)+m_i}{2m_i}\frac{E_j(\mathbf{k}_2)+m_j}{2m_j}}\cdot\\
&\quad\times\chi^\dagger_{-s}\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_2\boldsymbol{\sigma}}{E_j(\mathbf{k}_2)+m_j}+\frac{\boldsymbol{\sigma}\boldsymbol{\sigma}\cdot\mathbf{k}_1}{E_i(\mathbf{k}_1)+m_i}\right]\chi_r\\
&\approx \frac{(-)^{1/2-s}}{2m_Q}\chi^\dagger_{-s}\left[(\mathbf{k}_1+\mathbf{k}_2)+\boldsymbol{\sigma}\times(\mathbf{k}_1-\mathbf{k}_2)\right]\chi_r,\\
\left[\bar{v}(\mathbf{k}_2,s)\ \gamma^0\gamma_5\ u(\mathbf{k}_1,r)\right] &\approx (-)^{1/2-s}\chi^\dagger_{-s}\left[1+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_2\boldsymbol{\sigma}\cdot\mathbf{k}_1}{4m_q^2}\right]\chi_r \approx (-)^{1/2-s}\delta_{r,-s},
\end{aligned} \tag{75}$$

where we used the non-relativistic approximation, etc. We note that for spin-1 the $\sum_{s,r=\pm}$ the first term in the brackets [....] gives zero, and we get

$$\sum_{r,s=\pm}\chi^{(1)}_{-m}(r,s)\ \left[\bar{v}(\mathbf{k}_2,s)\ \boldsymbol{\gamma}\gamma_5\ u(\mathbf{k}_1,r)\right] = \frac{\sqrt{2}}{2m_q}\boldsymbol{\epsilon}\times\boldsymbol{\Delta}_M, \tag{76}$$

where $\boldsymbol{\Delta}_M = \mathbf{k}_2 - \mathbf{k}_1$.

To proceed, we have to evaluate [13]

$$C^{1\ 1\ 1}_{m_L m_\sigma m}\widetilde{\psi}_{M,m_L}\chi^{(1)}_{m_\sigma} \Rightarrow C^{1\ 1\ 1}_{m_L m_\sigma m}\Delta_{M,m_L}\left(\boldsymbol{\epsilon}\times\boldsymbol{\Delta}_M\right)_{m_\sigma} =$$
$$C^{1\ 1\ 1}_{m_L m_\sigma m}C^{1\ 1\ 1}_{m'_\sigma m'_L m_\sigma}\epsilon_{m'_\sigma}\ \Delta_{M,m'_L}\Delta_{M,m_L} \Rightarrow$$
$$4R_M^{-2}\ (-)^{m_L}\delta_{m'_L,-m_L}\ C^{1\ 1\ 1}_{m_L m_\sigma m}C^{1\ 1\ 1}_{m'_\sigma m'_L m_\sigma}\epsilon_{m'_\sigma} =$$
$$4R_M^{-2}(-)^{m_L}\ C^{1\ 1\ 1}_{m_L m_\sigma m}C^{1\ \ 1\ \ 1}_{m'_\sigma -m_L m_\sigma}\epsilon_{m'_\sigma} = -4R_M^{-2}\epsilon_m.$$

Here, the second $\Rightarrow$ marks the performance $\int d^3\Delta_M$, and we used the relations

$$\begin{aligned} C^{1\ 1\ 1}_{m'_\sigma m'_L m_\sigma} &= -C^{1\ 1\ 1}_{m'_L m'_\sigma m_\sigma} = (-)^{m'_L}C^{1\ \ 1\ \ 1}_{m'_L -m_\sigma -m'_\sigma} \\ &= -(-)^{m'_L}C^{\ 1\ \ 1\ \ 1}_{-m'_L m_\sigma m'_\sigma} \Rightarrow -(-)^{m_L}C^{1\ 1\ 1}_{m_L m_\sigma m'_\sigma}. \end{aligned}$$

Using these results, we get for the meson-overlap matrix element (73)

$$\langle 0|\bar{q}_i(\mathbf{x})\boldsymbol{\gamma}\gamma_5\ q_j(\mathbf{x})|M_A(\mathbf{k},m)\rangle = +\frac{2}{m_Q R_M}\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$$
$$\times(\mathbf{k}_1-\mathbf{k}_2)^2\cdot(2\pi)^{-3}\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_1,\mathbf{k}_2)\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right]\ \boldsymbol{\epsilon}_m. \tag{77}$$

(ii) Baryon-overlap matrix element: The quark-transition matrix element for the scalar operator is

$$\langle q_\beta(\mathbf{k}_4)|\bar{q}_j(\mathbf{x})\boldsymbol{\gamma}\gamma_5\ q_i(\mathbf{x})|q_\alpha(\mathbf{q}_3)\rangle = (2\pi)^{-3}\sqrt{\frac{E_j(\mathbf{k}_4)+m_j}{2m_j}\frac{E_i(\mathbf{k}_3)+m_i}{2m_i}}\ \cdot$$
$$\times\chi_j\ \dagger\left\{\boldsymbol{\sigma}+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_4}{E_j(\mathbf{k}_4)+m_j}\boldsymbol{\sigma}\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_3}{E_i(\mathbf{k}_3)+m_i}\right\}\chi_i\ \exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big]$$
$$\approx(2\pi)^{-3}\delta_{i\alpha}\delta_{j\beta}\ \boldsymbol{\sigma}\ \exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big], \tag{78}$$

where again we used the non-relativistic approximation. Folding this matrix element, we get

$$\langle B(\mathbf{p}')|\bar{q}_j(\mathbf{x})\ \boldsymbol{\gamma}\gamma_5\ q_i(\mathbf{x})|B(\mathbf{p})\rangle = \delta_{ij}(2\pi)^{-3}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\ \cdot$$
$$\times\int d^3k'_3 d^3k_4\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot\widetilde{\psi}^*_B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \cdot$$
$$\times\exp\Big[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\Big]\cdot\Big[\chi^\dagger_{B,s_f}\ \boldsymbol{\sigma}_i\ \chi_{B,s_i}\Big]. \tag{79}$$

Combining factors, and summing over the three quarks of the baryon $\sum_{i=1,3}$, we get finally for the transition matrix

$$\langle B(\mathbf{p}'),M_A(\mathbf{k},m)|H_{int}|B(\mathbf{p})\rangle = \int d^3x\ \langle B(\mathbf{p}'),M_A(\mathbf{k},m)|\mathcal{H}_{int}(\mathbf{x})|B(\mathbf{p})\rangle$$
$$\Rightarrow -(2\pi)^{-3}\frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}}\ \frac{6}{M_B R_M}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\ \cdot$$
$$\times\int d^3k'_3 d^3k_4 d^3k_5\delta(\mathbf{k_3}-\mathbf{k'_3}-\mathbf{k_4}-\mathbf{k_5})\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\ \cdot$$
$$\times\widetilde{\psi}^*_B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}'_3,\mathbf{k}_5)^*.\chi^\dagger_{s_f}\Big[\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_m\Big]\chi_{s_i}, \tag{80}$$

where the (-)-sign in front is taken from the Fierz-identity in the $^3P_0$-model. From Appendix B it is easily seen that the overlap-integral in (80) is given by

$$I_0^{dir}(A;B,M) \xrightarrow{\mathbf{k}=0} 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} .$$

The axial-vector BBM-coupling is, apart from the spinor factor $\chi_{s_f}^\dagger\,[\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}_m]\,\chi_{s_i}$, given by the matrix element

$$\begin{aligned}\langle B(\mathbf{0}'), M_{A,m}(\mathbf{0})|H_{int}|B(\mathbf{0})\rangle &\sim -\frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{6}{M_B R_M}\cdot(2\pi)^{-3}\cdot \\ \times\ 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} &- (2\pi)^{-3}\frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{12\sqrt{2}}{M_B R_M}\left(\frac{\pi}{R_M^2}\right)^{3/4} . \end{aligned} \tag{81}$$

The hadron level interaction Hamiltonian density for the ω-meson to the baryons is

$$\mathcal{H}_I = g_A[\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)]\ A^\mu(x). \tag{82}$$

giving the matrix element

$$\begin{aligned}\langle E_1(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle &= \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\omega(\mathbf{k})}}\sqrt{\frac{M^2}{E_{p'}E_p}}\cdot g_A\cdot \\ \times\left[\bar{u}(\mathbf{p}')\gamma_\mu\gamma_5 u(\mathbf{p})\cdot\epsilon^\mu(\mathbf{k}_M)\right]&\ \delta(\mathbf{p}-\mathbf{p}'-\mathbf{k})\ . \end{aligned} \tag{83}$$

Using the non-relativistic approximation (N.R.) (83) leads to

$$\begin{aligned}\langle \mathbf{E}_1(\mathbf{k}), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle\ &=\ \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2m_A}}\cdot g_A\ \chi_{s_z'}^\dagger\ \{\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\}\,\chi_{s_z}\cdot \\ &\times\delta(\mathbf{p}-\mathbf{p}'-\mathbf{k}). \end{aligned} \tag{84}$$

From (84) etc. one obtains upon the comparison of (84) with (81) for the NNE_1-coupling, including the isospin factor $1/\sqrt{2}$, the expression

$$\begin{aligned} g_A\ &=\ -(2\pi)^{-3}\cdot\frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2}\frac{12\sqrt{2}}{M_B R_M}\left(\frac{\pi}{R_M^2}\right)^{3/4}(2\pi)^{3/2}\sqrt{2m_A}/\sqrt{2} \\ &=\ -\pi^{-3/4}\gamma_{q\bar{q}}\ \frac{\sqrt{m_A R_M}}{(\Lambda_{QPC}R_M)^2}\frac{6}{M_B R_M}\sim R_M^{-5/2}. \end{aligned} \tag{85}$$

Comparing this with (39) we find for the ratio

$$g_A/g_S = -\frac{2}{3}\sqrt{\frac{m_A}{m_S}}. \tag{86}$$

For $m_A = 1270$, $m_S = 750$ MeV, and $R_M = 0.7$ fm, we have $g_A/g_S = -0.79$, which is more or less what can be expected naively from the Fierz-identities in the $^3P_0$-model.

F. Axial-vector mesons, II

NOTE: (i) From the relation $k \cdot \rho_B = k_0\rho_{B,0} - \mathbf{k}\cdot\boldsymbol{\rho}_B = 0$, where $\rho_B^\mu(\lambda)$ is the B_1 polarization vector, one infers that under the parity transformation $\rho_{B,0}$ is pseudoscalar-like. Then, in first instance one would be inclined to treat $\rho_{B,0}$ as a $^1S_0$ $q\bar{q}$-state! Consequently, the NR-treatment seems similar to the pseudoscalar-meson case. (ii) However the pseudoscalar and axial-vector states behave differently under charge-conjugation. From $C\boldsymbol{\rho_B}(\mathbf{k}) = -\boldsymbol{\rho_B}(\mathbf{k})$, it follows that also $C\rho_0(\mathbf{k}) = -\rho_0(\mathbf{k})$, and therefore can not be represented by a $^1S_0$-state which has C=+1. Therefore, we have to use the $\boldsymbol{\rho_B}(\mathbf{k})$-coupling. This leads, see below, to zero coupling for the axial B-mesons.

(i) Axial-meson-overlap matrix element:

$$\langle 0|\bar{q}_i(x)\gamma_\mu\gamma_5\ q_j(x)|M_B(\mathbf{k},m)\rangle = \sum_{r,s=\pm}\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\cdot$$
$$\times\widetilde{\psi}^{(L=1)}_{M,m}(\mathbf{k}_1,\mathbf{k}_2)\cdot\chi^{(0)}(r,s)\ \langle 0|\bar{q}_i(x)\gamma^\mu\gamma_5\ q_j(x)\ b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle. \tag{87}$$

The vacuum expectation is given as

$$\langle 0|\bar{q}_i(\mathbf{x})\gamma^\mu\gamma_5\ q_j(\mathbf{x})\ b^\dagger_\alpha(\mathbf{k}_1,r)d^\dagger_\beta(\mathbf{k}_2,s)|0\rangle = \delta_{i\beta}\delta_{j\alpha}(2\pi)^{-3}\sqrt{\frac{m_i}{E_i(k_1)}\frac{m_j}{E_j(k_2)}}\cdot$$
$$\times\left[\bar{v}(\mathbf{k}_2,s)\ \gamma^\mu\gamma_5\ u(\mathbf{k}_1,r)\right]\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right]. \tag{88}$$

The Dirac bispinor matrix element gives, for the space and zero components,

$$\left[\bar{v}(\mathbf{k}_2,s)\ \boldsymbol{\gamma}\gamma_5\ u(\mathbf{k}_1,r)\right] \approx \frac{(-)^{1/2-s}}{2m_Q}\chi^\dagger_{-s}\left[(\mathbf{k}_1+\mathbf{k}_2)+\boldsymbol{\sigma}\times(\mathbf{k}_1-\mathbf{k}_2)\right]\chi_r,$$
$$\left[\bar{v}(\mathbf{k}_2,s)\ \gamma^0\gamma_5\ u(\mathbf{k}_1,r)\right] \approx (-)^{1/2-s}\delta_{r,-s}, \tag{89}$$

where we used the non-relativistic approximation, etc. We note that for spin-1 the $\sum_{s,r=\pm}$ the first term in the brackets [....] gives zero, and we get

$$\sum_{r,s=\pm}\chi^{(0)}(r,s)\ \left[\bar{v}(\mathbf{k}_2,s)\ \gamma^0\gamma_5\ u(\mathbf{k}_1,r)\right] = -\sqrt{2},$$
$$\sum_{r,s=\pm}\chi^{(0)}(r,s)\ \left[\bar{v}(\mathbf{k}_2,s)\ \boldsymbol{\gamma}\gamma_5\ u(\mathbf{k}_1,r)\right] = -\frac{\sqrt{2}}{2m_Q}\ \mathbf{k}_M. \tag{90}$$

Using these results, we get for the meson-overlap matrix element (87)

$$\langle 0|\bar{q}_i(\mathbf{x})\gamma^0\gamma_5\ q_j(\mathbf{x})|M_B(\mathbf{k},m)\rangle = -iR_M\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$$
$$\times(\mathbf{k}_1-\mathbf{k}_2)_m\cdot(2\pi)^{-3}\ \widetilde{\psi}^{(L=0)}_M(\mathbf{k}_1,\mathbf{k}_2)\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right]\ \boldsymbol{\epsilon}_m,$$
$$\langle 0|\bar{q}_i(\mathbf{x})\boldsymbol{\gamma}\gamma_5\ q_j(\mathbf{x})|M_B(\mathbf{k},m)\rangle = +\frac{2}{m_Q R_M}\int d^3k_1 d^3k_2\ \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$$
$$\times(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{k}_M\cdot(2\pi)^{-3}\ \widetilde{\psi}^{(L=0)}_M(\mathbf{k}_1,\mathbf{k}_2)\ \exp\left[+i\left(\mathbf{k}_1+\mathbf{k}_2\right)\cdot\mathbf{x}\right]\ \boldsymbol{\epsilon}_m. \tag{91}$$

(ii) Baryon-overlap matrix element: The quark-transition matrix element for the axial-vector operator is

$$\langle q_\beta(\mathbf{k}_4)|\bar{q}_j(\mathbf{x})\gamma^0\gamma_5\ q_i(\mathbf{x})|q_\alpha(\mathbf{q}_3)\rangle = (2\pi)^{-3}\sqrt{\frac{E_j(\mathbf{k}_4)+m_j}{2m_j}\frac{E_i(\mathbf{k}_3)+m_i}{2m_i}}\ \cdot$$
$$\times\chi_j\dagger\left\{\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_4}{E_j(\mathbf{k}_4)+m_j}+\frac{\boldsymbol{\sigma}\cdot\mathbf{k}_3}{E_i(\mathbf{k}_3)+m_i}\right\}\chi_i\ \exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right]$$
$$\approx(2\pi)^{-3}\frac{\delta_{ij}}{2m_Q}\ \left[\chi_j^\dagger\ \boldsymbol{\sigma}\cdot(\mathbf{k}_3+\mathbf{k}_4)\,\chi_i\right]\cdot\exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right],$$
$$\langle q(\mathbf{k}_4)|\bar{q}_j(\mathbf{x})\boldsymbol{\gamma}\gamma_5\ q_i(\mathbf{x})|q(\mathbf{q}_3)\rangle\approx(2\pi)^{-3}\delta_{i\alpha}\delta_{j\beta}\ \boldsymbol{\sigma}\ \exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right], \tag{92}$$

where again we used the non-relativistic approximation.

As explained in the beginning of this subsection, we must use the space-component of the B-vector meson. Folding this matrix element, we get

$$\langle B(\mathbf{p}')|\bar{q}_j(\mathbf{x})\ \boldsymbol{\gamma}\gamma_5\ q_i(\mathbf{x})|B(\mathbf{p})\rangle=\delta_{ij}(2\pi)^{-3}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\ \cdot$$
$$\times\int d^3k_3'd^3k_4\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot\widetilde{\psi}^*_B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \cdot$$
$$\times\exp\left[i(\mathbf{k}_4-\mathbf{k}_3)\cdot\mathbf{x}\right]\cdot\left[\chi^\dagger_{B,s_f}\ \boldsymbol{\sigma}_i\ \chi_{B,s_i}\right]. \tag{93}$$

Combining factors, and summing over the three quarks of the baryon $\sum_{i=1,3}$, we get finally for the transition matrix

$$\langle B(\mathbf{p}'),M_B(\mathbf{k},m)|H_{int}|B(\mathbf{p})\rangle=\int d^3x\ \langle B(\mathbf{p}'),M_B(\mathbf{k},m)|\mathcal{H}_{int}(\mathbf{x})|B(\mathbf{p})\rangle$$
$$\Rightarrow +i(2\pi)^{-3}\frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}}\frac{3R_M}{2M_B}\int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\ \cdot$$
$$\times\int d^3k_3'd^3k_4d^3k_5\delta(\mathbf{k_3}-\mathbf{k_3'}-\mathbf{k_4}-\mathbf{k_5})\ \delta(\mathbf{p}'-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\ \cdot$$
$$\times\widetilde{\psi}^*_B(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_3',\mathbf{k}_5)^*\left((\mathbf{k}_3'-\mathbf{k}_5)\cdot\mathbf{k}_M\right)\cdot$$
$$\times\left[\chi^\dagger_{s_f}\boldsymbol{\sigma}\chi_{s_i}i\right], \tag{94}$$

where the (-)-sign from the Fierz-identity in the $^3P_0$-model is taken into account. **From Appendix B it is easily seen that the overlap-integral in (94) is zero!**

The hadron level interaction Hamiltonian density

$$\mathcal{H}_I=\frac{if_{BNN}}{m_\pi}\left[\bar{\psi}\sigma_{\mu\nu}\gamma_5\boldsymbol{\tau}\psi\right]\cdot\partial^\nu\mathbf{B}_1^\mu\ , \tag{95}$$

giving, apart from the isospin factor, the matrix element

$$\langle B_1(\mathbf{k}\mu),N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle=\sqrt{\frac{1}{2\omega(\mathbf{k})(2\pi)^3}}\sqrt{\frac{M^2}{E_{p'}E_p}}\ \cdot$$

$$\times \frac{f_{NNB_1}}{m_\pi} \left[\bar{u}(\mathbf{p}')\sigma_{\mu\nu}\gamma_5 u(\mathbf{p}) \cdot (p-p')^\nu \epsilon^\mu(\mathbf{k}) \right] \cdot$$
$$\times (2\pi)\delta(E_f - E_i)\ \delta(\mathbf{p}' - \mathbf{p} - \mathbf{k})\ . \tag{96}$$

The expression betweem square brackets [...] with $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$ [11], is

$$\left[\bar{u}(\mathbf{p}')\sigma_{\mu\nu}\gamma_5 u(\mathbf{p}) \cdot (p-p')^\nu \epsilon^\mu(\mathbf{k}) \right] = -i(p'+p)\cdot\epsilon\ [\bar{u}(\mathbf{p}')\gamma_5 u(\mathbf{p})].$$

Using the non-relativistic approximation (N.R.) (96) leads to

$$\langle B_1(\mathbf{k}, \mu = 0), N'(\mathbf{p}')|H_I|N(\mathbf{p})\rangle = -\frac{i}{(2\pi)^{3/2}} \frac{f_{NNB_1}}{m_\pi\sqrt{2m_{B_1}}} \frac{(p'+p)\cdot\epsilon}{2M_B}\ \cdot$$
$$\times \left(\chi^\dagger_{s'_z}\ (\boldsymbol{\sigma}\cdot\mathbf{k})\ \chi_{s_z} \right)\ \delta(\mathbf{p} - \mathbf{p}' - \mathbf{k})\ (2\pi)\delta(E_f - E_i)\ . \tag{97}$$

G. Tensor-mesons

In the QPC-model there is no coupling for the tensor-mesons ($J^{PC} = 2^{++}$). In the case of the $^3P_0$ interaction Hamiltonian (11) there occurs a term with the antisymmetric-tensor bilinear, which has $J^{PC} = 2^{+-}$, see e.g. [14], paragraph 3-4-4. However, such quantum numbers do not correspond to a $q\bar{q}$-state, see [15].

To generate non-zero symmetric tensor-meson coupling one could introduce the QPC-interaction with a new phenomenological parameter $\gamma^{(T)}_{q\bar{q}}$ as follows:

$$\mathcal{H}^{(T)}_I = -\gamma^{(T)}_{q\bar{q}}\ [\bar{q}_i\gamma^\mu\partial^\nu q_i]\cdot[\bar{q}_j\gamma_\mu\partial_\nu q_j] =$$
$$+\gamma^{(T)}_{q\bar{q}} \sum_{i,j} \left[+\ \bar{q}_i\ \partial^\nu q_j \cdot \bar{q}_i\ \partial_\nu q_j - \frac{1}{2}\ \bar{q}_i\gamma^\mu\partial^\nu q_j\cdot\bar{q}_j\gamma^\mu\partial_\nu q_i \right.$$
$$\left. -\frac{1}{2}\ \bar{q}_i\gamma^\mu\gamma_5\partial^\nu q_j\cdot\bar{q}_j\gamma^\mu\gamma^5\partial_\nu q_i -\ \bar{q}_i\gamma_5\partial^\nu q_j\cdot\bar{q}_j\gamma^5\partial_\nu q_i \right]. \tag{98}$$

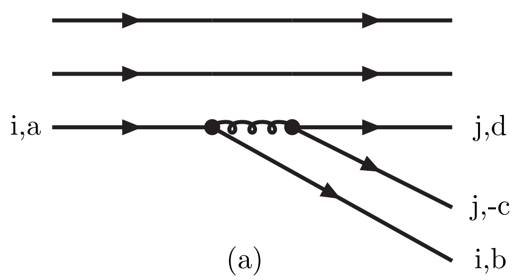

FIG. 2: *Isospin labels QPC-model*

IV. ISOSPIN FACTORS

In Fig. 2 the isospin labels of the active quarks in the creation process are shown. Since we consider here **the neutral-meson coupling**, there is no-flavor change in this proces for the quarks of the baryons, there is a factor δ_{ij} in the matrix elements (see above). The isospin wave functions of the $q\bar{q}$-state for $I_M = 0$ and $I_M = 1$ they are

$$\varphi^{(0)}(c,d) \;=\; \left(\delta_{c,+1/2}\delta_{d,-1/2} - \delta_{c,-1/2}\delta_{d,+1/2}\right)/\sqrt{2}, \tag{99}$$

$$\varphi^{(1)}_n(c,d) \;=\; \begin{cases} \delta_{c,+1/2}\delta_{d,+1/2} & n=+1 \\ \left(\delta_{c,+1/2}\delta_{d,-1/2} + \delta_{c,-1/2}\delta_{d,+1/2}\right)/\sqrt{2} & n=0 \\ \delta_{c,-1/2}\delta_{d,-1/2} & n=-1 \end{cases} \tag{100}$$

Notice that we put the antiquark in second place, and the spin-up/down state for the antiquark has $s = \pm 1/2$.

For the isospin description of the antiquark we note that the charge-conjugate Dirac wave function is [11]

$$\psi_C(x) \;=\; \eta_C C \bar{\psi}^T(x), \;\; C = -i\gamma_2\gamma_0, \;\eta_C = 1.$$

Evidently, $\psi_C(x) \sim \psi^*(x)$ and complex conjugation is involved. The complex conjugation (c.c) of the τ-commutator relations gives

$$[\tau_i, \tau_j] = i\epsilon_{ijk}\;\tau_k \xrightarrow{c.c} [-\tau_i^*, -\tau_j^*] = -i\epsilon_{ijk}\;(-\tau_k^*),$$

and therefore to obtain the proper isospin spinors we have to find in isospin space a unitary matrix A such that $A\tau_i A^{-1} = -\tau_i^*$ for the transformation of the quark isospinors to the proper antiquark isospinors. This leads to $A = \pm i\tau_2$. Choosing the (-)-sign gives for the isospin anti-spinors

$$\bar{\varphi}_a = A\varphi_a = (-)^{a-1/2}\varphi_{-a}.$$

From the Fierz-identities [2] the isospin matrix element is given as

$$\delta^a_b\;\delta^c_d \;=\; \frac{1}{2}\delta^a_d\;\delta^c_b + \frac{1}{2}\;\boldsymbol{\tau}^a_d\cdot\boldsymbol{\tau}^c_b. \tag{101}$$

1. For $I_M = 0$ we get

$$\Rightarrow \sum_{i=1}^{3}\cdot\sum_{b,c=\pm}(-)^{1/2-c}\varphi^{(0)}(b,c)\cdot\frac{1}{2}\delta^a_d\;\delta^c_b = -(3/\sqrt{2})\;\delta^a_d. \tag{102}$$

2. For $I_M = 1$ we get

$$\Rightarrow \sum_{i=1}^{3}\cdot\sum_{b,c=\pm}(-)^{1/2-c}\varphi^{(1)}(b,c)\cdot\frac{1}{2}\boldsymbol{\tau}^a_d\cdot\boldsymbol{\tau}^c_b = -(1/\sqrt{2})\;(\boldsymbol{\tau}_B)^a_d, \tag{103}$$

where we have summed over the quarks $\sum_{i=1,2,3}$, and $\boldsymbol{\tau}_B = \sum_{i=1,2,3}\boldsymbol{\tau}_i$, i.e. a sum over the quark isospin operators.

This settles that the couplings of the isosinglet mesons is a factor 3 larger than those for the isotriplet mesons.

The extension to flavor SU(3) is straightforward, using e.g. the results in [16] and references therein. The relations of the 9j-symbols in [16] and [18] is given in the note [17].

V. RESULTS, DISCUSSION AND CONCLUSIONS

Fortunately, the constants f_π, f_V, f_A, and f_S which appear in the C.F.I.'s seem to be determined pretty certainly.
The QPC-models have much predictive power. This because here a single phenomenological parameter, the pair-creation constant $\gamma_{q\bar{q}}$, predicts for example many ratio's of couplings. So, many relations are obtained in this model.

Vector-mesons: In [8] one came to the conclusion that at the origin the $q\bar{q}$ wave-function satisfies

$$|\psi_\pi(0)|^2 = \frac{1}{2} m_\pi^3 \ , \quad |\psi_V(0)|^2 = \frac{m_V}{m_\pi} |\psi_\pi(0)|^2 = \frac{1}{2} m_V m_\pi^2 \ . \tag{1}$$

Equation (1) and the N.R. wave-function, see e.g. [3], yields

$$|\psi_V(0)|^2 = \left(\pi R_M^2\right)^{-3/2} \quad \Rightarrow (m_V R_V)^3 = 2\pi^{-3/2} \left(\frac{m_V}{m_\pi}\right)^2 \ . \tag{2}$$

For the ρ-meson, we get from (2)

$$m_\rho R_\rho = 2.215 \ , \quad R_\rho = 0.57 \ fm \ .$$

From [3], eq. (3.15), we have

$$\begin{aligned} f_\rho &= f_{\rho\pi\pi} = f_{\rho NN} \approx \gamma \left(\frac{2}{3\pi}\right)^{1/2} \frac{m_\rho^{3/2}}{|\psi_\rho(0)|} \\ &= \frac{2\gamma}{\sqrt{3\pi}} \frac{m_\rho}{m_\pi} \ . \end{aligned} \tag{3}$$

On the other hand, from $\rho^0 \rightarrow e^+e^-$, one has [3], eq.(2.18) and eqn. (3.15),

$$f_\rho = \frac{m_\rho^{3/2}}{\sqrt{2}|\psi_\rho(0)|} \approx \gamma_0 \left(\frac{2}{3\pi}\right)^{1/2} \frac{m_\rho^{3/2}}{|\psi_\rho(0)|} \Rightarrow \gamma_0 = \frac{1}{2}\sqrt{3\pi} = 1.535 \ . \tag{4}$$

From [3], eqn. (3.15) and the N.R. wave-function eqn. (3.9), we have

$$\begin{aligned} f_\rho &= \gamma_0 \frac{4}{\sqrt{3\pi}} \left(m_\rho R_\rho\right)^{3/2} \ \Rightarrow \\ \frac{\gamma_0^2}{4\pi} &= \frac{3\pi}{16} \left(m_\rho R_\rho\right)^{-3} \ \frac{f_\rho^2}{4\pi} \ , \end{aligned} \tag{5}$$

which gives, using $f_\rho^2/4\pi = 2.4$, that $\gamma_0 = 1.55$ ($R_\rho = 0.5fm$), and $\gamma_0 = 3.34$ ($R_\rho = 0.3fm$).

Gluonic correction: Since in the QPC-model we used Gaussian wave-function, the $q\bar{q}$-potential is of the harmonic-oscillator type. This potential does not account for the $1/r$-behavior at short distance, which is due to the one-gluon-exchange potential. This implies that (4) should read

$$f_\rho = \frac{m_\rho^{3/2}}{\sqrt{2}|\psi_\rho(0)|} = \gamma\left(\frac{2}{3\pi}\right)^{1/2} \frac{m_\rho^{3/2}}{|'\psi_\rho(0)'|} \Rightarrow \gamma = \gamma_0 \frac{|'\psi_\rho(0)'|}{|\psi_\rho(0)|} \ , \tag{6}$$

where $'\psi_\rho(0)'$ is the harmonic-oscillator (h.o.) wave function, and $\psi_\rho(0)$ the true wave function. To first order we have [5, 19]

$$|\psi_\rho(0)| = |'\psi_\rho(0)'| \left(1 - \frac{16}{3}\frac{\alpha_s(m_\rho^2)}{\pi}\right)^{1/2} \ . \tag{7}$$

Then, for $\alpha_s(m_\rho^2) = 0.5$, we find that

$$\gamma = \gamma_0 \left(1 - \frac{16}{3}\frac{\alpha_s(m_\rho)}{\pi}\right)^{-1/2} \Rightarrow 3.94 \ , \tag{8}$$

which is close to the value 3.85 used in [5]. In Table I the relation (8) is shown. Here, we used from [15] the parameterization

$$\alpha_s(\mu) = 4\pi / \left(\beta_0 \ln(\mu^2/\Lambda_{QPC}^2)\right) \ , \tag{9}$$

with $\Lambda_{QPC} = 100$MeV and $\beta_0 = 11 - \frac{2}{3}n_f -> 9$. From this table one sees that at the scale

TABLE I: Pair-creation constant γ as function of α_s.

μ [GeV]	$\alpha_s(\mu)$	$\gamma(\mu)$
∞	0.00	1.535
80.0	0.10	1.685
35.0	0.20	1.889
1.05	0.30	2.191
0.55	0.40	2.710
0.40	0.50	3.94
0.35	0.55	5.96

of 1 GeV a value $\gamma = 2.19$ is reasonable. This value we will use later when comparing the QPC-model predictions and the ESC04-model coupling constants.

The γ-value for QPC-model: Identifying $g_V =: f_{NN\rho}/2$ we compare (5) and (50) we have, including the isospin factor $1/\sqrt{2}$,

$$\gamma \frac{2}{\sqrt{3\pi}} (m_V R_M)^{3/2} = \frac{3}{\sqrt{2}} \pi^{-3/4} \gamma_{q\bar{q}} \frac{\sqrt{m_V R_M}}{(\Lambda_{QPC} R_M)^2}. \tag{10}$$

This yields

$$\gamma_{q\bar{q}}/\gamma = \frac{2}{3}\sqrt{\frac{2}{3}}\, \pi^{1/4}\, (\Lambda_{QPC} R_M)^2\, (m_V R_M). \tag{11}$$

Using $\Lambda_{QPC} = 250$ MeV, $m_V = 750$ MeV, and $R_M = 0.5$ fm one obtains $\gamma_{q\bar{q}}/\gamma = 0.55$. For $R_M = 0.67$ fm the latter ratio becomes 1.0. We conclude: **the quark-antiquark pair creation constant in the QPC-model is fully compatible with the quark-annihilation process for $\rho^0 \longrightarrow e^+e^-$.**

A. Comparison QPC-predictions and ESC-fit NN

First, we summarize the formulas for the divers couplings for the I=1 mesons:

$$\begin{aligned}
g_P &= +\pi^{-3/4}\, \gamma_{q\bar{q}}\, \frac{(m_P R_M)^{1/2}}{(\Lambda_{QPC} R_M)^2} \cdot (6\sqrt{2})\ , \\
g_V &= +\pi^{-3/4}\, \gamma_{q\bar{q}}\, \frac{(m_V R_M)^{1/2}}{(\Lambda_{QPC} R_M)^2} \cdot (3/\sqrt{2})\ , \\
g_S &= +\pi^{-3/4}\, \gamma_{q\bar{q}}\, \frac{(m_S R_M)^{-1/2}}{(\Lambda_{QPC} R_M)^2} \cdot \frac{9 m_S}{M_B}\ , \\
g_A &= -\pi^{-3/4}\, \gamma_{q\bar{q}}\, \frac{(m_S R_M)^{-1/2}}{(\Lambda_{QPC} R_M)^2} \cdot \frac{6 m_A}{M_B}.
\end{aligned}$$

Writing the expressions, obtained in these notes, for the couplings in the following concise form:

$$g_{BBM}(\pm) = \gamma_{q\bar{q}} \frac{3}{\sqrt{2}}\, \pi^{-3/4}\, X_M\, (I_M, L_M)\ F_M^{(\pm)} \tag{12}$$

where $\pm = -(-)^{L_f}$ with L_f is the orbital angular momentum of the final MB-state, X_M is the product of the recoupling coefficients, and where

$$\begin{aligned}
F^{(+)} &= \frac{3}{2}\sqrt{2}\ (m_M R_M)^{1/2}\, (\Lambda_{QPC} R_M)^{-2}\ , \\
F^{(-)} &= \frac{3}{2}\sqrt{2}\ (m_M R_M)^{-1/2}\, (\Lambda_{QPC} R_M)^{-2} \cdot 3\sqrt{2}\ (M_M/M_B).
\end{aligned} \tag{13}$$

We note that there can be thought of various corrections to the (naive) QPC-formulas in (12): (i) isospin-dependent gluon-exchange corrections to the pair-creation process, similarly to those for non-leptonic weak decays in connention with the $\Delta T = 1/2$-rule; (ii) hadronic vertex corrections. Instead of trying to

TABLE II: ESC08c Couplings and $^3P_0$- and $^3S_1$-model relations, with ideal mixing for vector and scalar meson nonets (R_N = 0.54 fm). Pseudoscalar- and axial-nonets the mixing angles are -23^0 and -42.7^o respectively. We defined $\widetilde{X}_M$, i.e. the isospin factor. Here, Λ_{QPC} = 350 MeV, $\gamma(\alpha_s = 0.30) = 2.19$ etc. The weights are $A = 1.067$ and $B = 0.526$ for the $^3P_0$ and $^3S_1$ respectively.

Meson	$r_M[fm]$	$\widetilde{X}_M$	γ_M	$^3S_1$	$^3P_0$	QPC	ESC08c
$\pi(140)$	0.46	2	6.89	$g = -3.17$	$g = 6.40$	3.24	3.63
$\eta'(957)$	0.66	6	2.22	$g = -4.46$	$g = 9.05$	4.58	2.28
$\rho(770)$	0.66	1	2.37	$g = -0.37$	$g = 1.49$	1.13	0.69
$\omega(783)$	0.66	3	2.35	$g = -1.11$	$g = 4.50$	3.39	3.52
$a_0(962)$	0.66	$3\sqrt{2}$	2.22	$g = 0.52$	$g = 1.06$	1.58	0.87
$\epsilon(760)$	0.66	$9\sqrt{2}$	2.37	$g = 1.49$	$g = 3.02$	4.51	4.74
$a_1(1270)$	0.66	$2\sqrt{2}$	2.09	$g = -0.19$	$g = -0.77$	-0.96	-1.11
$d_1(1280)$	0.66	$6\sqrt{2}$	2.09	$g = -0.64$	$g = -2.58$	-3.22	-1.06
$b_1(1235)$	0.66		2.19	$f = 0$	$f = 0$	0.0	-0.20
$a_2(1320)$	0.66		2.19	$g = 0$	$g = 0$	0.0	0.00

account for these effects using the QCD-corrections in a sophisticated comprehensive treatment we adjust the $q\bar{q}$-radii for each meson. We used a linear combination of the $^3P_0$- and the $^3S_1$-model with weights $A = 1.114$ and $B = 0.564$ respectively. We notice the dominance of the $^3P_0$- over the $^3S_1$-mechanism.

In the tables below, we show the results for the couplings of the $^3S_1,^3P_0$-model and compare them with the values obtained in a typical fit to the NN-data with the ESC-model. For the motivation for Λ_{QPC} = 350 MeV, $R_M = 0.66$, and $\gamma_M = 2.19$ see Appendix G. First, we show in Table II the results for 'naive' trial with identical radii for all vector-, scalar-, and axial-vector-mesons. Notice that the I=1 couplings are too strong.

We can produce many solutions with different Λ_{QPC}-values by noting the invariance of the

BBM-couplings under the scale-transformation:

$$R_M^+ = s\ R_M\ ,R_M^- = s^{3/5}\ R_M\ ,\ \Lambda_{QPC} = s^{-3/4}\Lambda_{QCD},$$

where $R_M^\pm$ refer to the mesons with $g_M \sim R_M^{-3/2}$ respectively $\sim R_M^{-5/2}$. In Table III we show this invariance explicitly starting from the 'naive' solution values for s=1 in Table II.

TABLE III: Scaling Λ_{QPC}, and the meson radii $R_M^\pm$ such as to keep $C^+ = \Lambda_{QPC}(R_M^+)^{3/4}$ and $C^- = \Lambda_{QPC}(R_M^-)^{5/4}$ invariant

s	Λ_{QPC} [MeV]	R_M^+ [fm]	R_M^- [fm]
1.00	350.0	0.660	0.860
0.90	378.8	0.594	0.807
0.80	413.8	0.528	0.752
0.70	457.3	0.462	0.694
0.60	513.4	0.396	0.633
0.50	588.6	0.330	0.567
0.40	695.9	0.264	0.496
0.30	863.4	0.198	0.418
0.20	1170.3	0.132	0.327
0.10	1968.2	0.066	0.216

Next, we show the two 'solutions': (i) Table IV: a solution with 66 % and 33 % for $^3P_0$ and $^3S_1$ respectively, and (ii) Table V: a solution with 99 % and 1 % for $^3P_0$ and $^3S_1$ respectively. Here, for this illustration we used a version of the ESC08c-model with the same mixing angles as used in ESC04-models. Here, no QQG form factor nor SU(6)-breaking is included.
In Table IX also the mixing anngles are as in ESC04-models, but now both the QQG form factor and SU(6)-breaking effects are included.

In Table VI we show the results for the final version of the ESC08c-model, having updated values for the mixing angles of the pseudoscalar and axial-vector mesons. The ESC08c-couplings and the QPC-couplings agree very well. In particularly, the SU(6)-breaking improves the agreement significantly. All this hints to the reality of the ESC08c couplings. An exception is the f_1(1420) coupling, which seems 'abnormal'. The ratio $g_{NNf_1}/g_{NNa_1} \approx 1$, instead of ≈ 3 as expected from the Quark-model. Probably this coupling is contaminated by the presence of the heavy pseudoscalar nonet around $\pi(1300)$ MeV. Enlarging the f_1(1420)-coupling and including its heavy pseudoscalar counterpart is a possible solution of this abnormality.

TABLE IV: ESC08c Couplings and ${}^3P_0$- and ${}^3S_1$-model relations, with ideal mixing for vector and scalar meson nonets (R_N = 0.54 fm). Pseudoscalar- and axial-nonets the mixing angles are -23^0 and -42.7^o respectively. We defined $\widetilde{X}_M$, i.e. the isospin factor. Here, Λ_{QPC} = 350 MeV, $\gamma(\alpha_s = 0.30) = 2.19$ etc. The weights are $A = 1.114$ and $B = 0.564$ for the ${}^3P_0$ and ${}^3S_1$ respectively.

Meson	$r_M[fm]$	$\widetilde{X}_M$	γ_M	${}^3S_1$	${}^3P_0$	QPC	ESC08c
$\pi(140)$	0.46	2	6.89	$g = -3.39$	$g = 6.68$	3.30	3.63
$\eta'(957)$	0.86	6	2.22	$g = -2.32$	$g = 4.58$	2.26	2.28
$\rho(770)$	0.86	1	2.37	$g = -0.27$	$g = 1.05$	0.78	0.69
$\omega(783)$	0.66	3	2.35	$g = -1.19$	$g = 4.70$	3.51	3.52
$a_0(962)$	0.86	$3\sqrt{2}$	2.22	$g = 0.29$	$g = 0.57$	0.86	0.87
$\epsilon(760)$	0.66	$9\sqrt{2}$	2.37	$g = 1.60$	$g = 3.16$	4.76	4.74
$a_1(1270)$	0.66	$2\sqrt{2}$	2.09	$g = -0.20$	$g = -0.80$	-1.00	-1.11
$f_1(1420)$	0.96	$6\sqrt{2}$	2.09	$g = -0.23$	$g = -0.91$	-1.14	-1.06
$b_1(1235)$	0.66		2.19	$f = 0$	$f = 0$	0.0	-0.20
$a_2(1320)$	0.66		2.19	$g = 0$	$g = 0$	0.0	0.00

The 'naive' predictions of the QPC-model are

$$\begin{array}{lll} g_\omega = 3g_\rho, & g_\epsilon = 3g_{a_0}, & \varepsilon_0(\lambda) \sim \bar{q}q({}^3P_0) \\ g_{a_0} \approx g_\rho, & g_\epsilon \approx g_\omega, & \varepsilon_a(\lambda) \sim \bar{q}q({}^3S_1) \end{array}$$

$$f_{NNa_1} \approx \frac{m_{a_1}}{m_\pi} f_{NN\pi} \approx 2.54 \quad \text{(CS)},$$

where for the f_{NNa_1} we quoted the prediction of Schwinger [23]. It is clear that these relations hold approximately only. In the ESC04-model the 'naive' relations were much better satisfied as in the ESC08c-model. In particularly, the Schwinger value [23] does not hold in ESC08c. As stressed in the foregoing paragraph, QPC-model gives in our view the so-called "bare" couplings (!) In Fig. 1 on the r.h.s. graph (a) indicates the QPC-model BBM-coupling prediction, and graph (b) the lowest order meson-exchange correction to the

TABLE V: ESC08c Couplings and $^3P_0$- and $^3S_1$-model relations, with ideal mixing for the vector and scalar meson nonets ($R_N = 0.54$ fm). Pseudoscalar- and axial-nonets the mixing angles are -23^0 and -42.7^o respectively. We defined $\widetilde{X}_M$, i.e. the isospin factor. Here, $\Lambda_{QPC} = 600$ MeV, $\gamma(\alpha_s = 0.30) = 2.19$ etc. The weights are $A = 0.865$ and $B = 0.004$ for the $^3P_0$ and $^3S_1$ respectively.

Meson	$r_M[fm]$	$\widetilde{X}_M$	γ_M	$^3S_1$	$^3P_0$	QPC	ESC08c
$\pi(140)$	0.30	2	6.89	$g = -0.02$	$g = 3.35$	3.33	3.63
$\eta'(957)$	0.56	6	2.22	$g = -0.01$	$g = 2.31$	2.30	2.28
$\rho(770)$	0.45	1	2.37	$g = -0.00$	$g = 0.73$	0.73	0.69
$\omega(783)$	0.33	3	2.35	$g = -0.01$	$g = 3.51$	3.51	3.52
$a_0(962)$	0.45	$3\sqrt{2}$	2.22	$g = 0.00$	$g = 0.76$	0.76	0.87
$\epsilon(760)$	0.33	$9\sqrt{2}$	2.37	$g = 0.02$	$g = 4.72$	4.74	4.74
$a_1(1270)$	0.33	$2\sqrt{2}$	2.09	$g = -0.00$	$g = -1.20$	-1.20	-1.11
$f_1(1420)$	0.51	$6\sqrt{2}$	2.09	$g = -0.00$	$g = -1.12$	-1.12	-1.06
$b_1(1235)$	0.56		2.19	$f = 0$	$f = 0$	0.0	-0.20
$a_2(1320)$	0.56		2.19	$g = 0$	$g = 0$	0.0	0.00

BBM-vertex. If we consider $I = 0$ and $I = 1$ exchange in this vertex, we have for example for the NN-couplings of the scalar mesons

$$\Delta g_{a_0} = \Delta_0 - 2\Delta_1 \quad , \quad \Delta g_{f_0} = \Delta_0 \ , \tag{14}$$

where $\Delta_{0,1}$ stands for the correction due to the $I = 0$ and $I = 1$ respectively. Then, such dressing corrections could bridge the gap between the QPC-predictions and the ESC04-fit for the coupling constants. For instance, if $\Delta_1 \approx (2/3)\Delta_0$, we would have $\Delta g_{a_0} \approx -(1/3)\Delta_0$ and $\Delta g_{f_0} = \Delta_0$. To work out a more or less complete model for these vertex corrections, which is by itself a rather elaborate program, is clearly beyond the scope of these notes.

TABLE VI: SU(6)-breaking in coupling constants, using (56) and (70)-irrep mixing with angle $\varphi = -22^o$ for the $^3P_0$- and $^3S_1$-model. Gaussian Quark-gluon cut-off $\Lambda_{QQG} = 986.6$ MeV. Ideal mixing for vector and scalar meson nonets. Pseudoscalar- and axial-nonets the mixing angles are -13^0 and $+50.0^o$ respectively, imposing the OZI-rule. Here, $\Lambda_{QPC} = 244.3$ MeV, $\gamma(\alpha_s = 0.30) = 2.19$ etc. The weights are A=0.677 and B=0.323 for the $^3P_0$ and $^3S_1$ respectively. The values in parentheses in the column QPC denote the results for $\varphi = 0^o$.

Meson	$r_M[fm]$	γ_M	$^3S_1$	$^3P_0$	QPC	ESC08c
$\pi(140)$	0.23	5.51	$g = -3.54$	$g = +7.40$	3.87 (4.07)	3.64
$\eta'(957)$	0.71	2.22	$g = -2.83$	$g = +5.93$	3.10 (3.72)	3.07
$\rho(770)$	0.71	2.37	$g = -0.24$	$g = +0.99$	0.75 (0.92)	0.73
$\omega(783)$	0.71	2.35	$g = -1.10$	$g = +4.60$	3.50 (3.45)	3.51
$a_0(962)$	0.81	2.22	$g = +0.28$	$g = +0.58$	0.86 (0.90)	0.89
$\epsilon(760)$	0.71	2.37	$g = +1.42$	$g = +2.96$	4.38 (4.37)	4.36
$a_1(1270)$	0.61	2.09	$g = -0.20$	$g = -0.84$	-1.05 (-1.06)	-1.10
$f_1(1420)$	0.61	2.09	$g = -0.72$	$g = -3.03$	-3.76 (-3.25)	-0.91

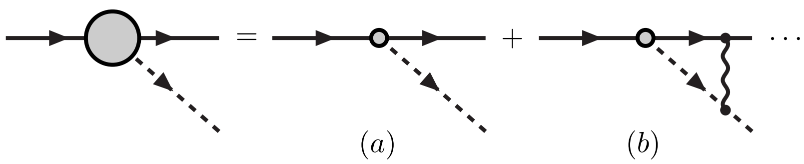

FIG. 1: *Vertex-dressing correction to QPC-coupling*

B. Isospin Dependence QPC Meson-couplings ?

It is well known that for weak non-leptonic transitions the hard-gluon corrections give a (partial) explanation of the $\Delta I = 1/2$-rule [2]. This comes about because of the correlation between isospin and color through the particle statistics. Likewise, in this subsection we

TABLE VII: Meson radii from the literature [20–22]

Meson	M(exp)/GeV	M(the)/GeV	$\sqrt{\langle r^2\rangle}$/fm	$\sqrt{\langle r^2\rangle}$/fm [21]	$\sqrt{\langle r^2\rangle}$/fm [22]
$\pi(1^1S_0)$	$\pi(140)$	0.139	0.337	0.512	0.64
$\eta(1^1S_0)$	$\eta'(957)$	—	—	—	—
$\rho(1^3S_1)$	$\rho(770)$	0.775	0.706	0.769	0.72
$\phi(1^3S_1)$	$\phi(1020)$	0.958	0.640	0.647	0.46
$\omega(1^3S_1)$	$\omega(783)$	—	—	—	0.72
$a_0(1^3P_1)$	$a_0(962)$	—	—	—	
$\epsilon(1^3P_1)$	$\epsilon(760)$	—	—	—	—
$a_1(1^3P_1)$	$a_1(1260)$	1.206	0.856	0.993	—
$d_1(1^3P_1)$	$d_1(1280)$	—	—	—	—
$b_1(1^1P_1)$	$b_1(1235)$	1.293	0.940	0.978	—

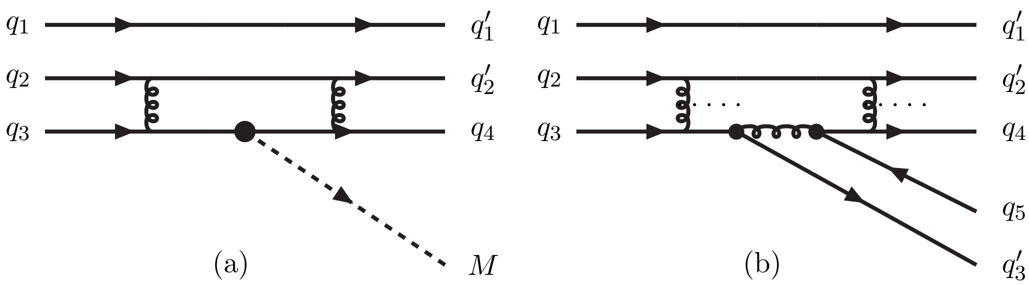

FIG. 2: *Gluonic-corrections BBM in QPC-models*

discuss the possibility for the isospin dependence of the BBM-couplings in the QPC-model. In Fig. 2 (multi) gluon-exchange corrections are depicted. We analyze here the corrections due to one-gluon exchange. The isospin for quarks 2 and 3 is given by $q_2q_3 = (q_2q_3 + q_3q_2)/2 + (q_2q_3 - q_3q_2)/2 \to |i_{23}\rangle = [|i_{23} = 1\rangle + |i_{23} = 0\rangle]/\sqrt{2}$. Combined with color, the

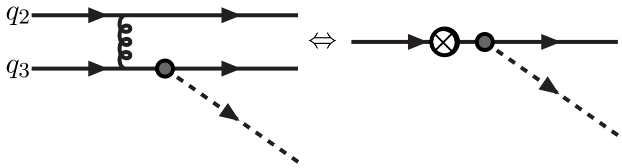

FIG. 3: *Isospin I_M dependent correction to QPC-coupling*

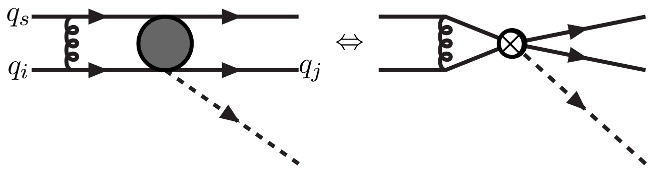

FIG. 4: *Hard-gluon correction to QPC-coupling*

isospin-color state has

$$|23\rangle_{I,C} \;=\; \frac{1}{\sqrt{2}}|i_{23}=1\rangle \otimes |c_{23}=3^*\rangle + \frac{1}{\sqrt{2}}|i_{23}=0\rangle \otimes |c_{23}=6\rangle$$

However, the 3 quarks in a baryon are in a color-singlet state. Therefore any subsystem of two quarks must be in the $\{3^*\}_c$-irrep. In our discussion we have to include also the flavor-symmetry. In SU(6) the $J^P=(1/2)^+$-states in the $\{56\}$-irrep the flavor-spin-color states are, see [24],

$$|(8,\mathbf{2})\rangle = \frac{1}{\sqrt{2}}\left[\phi_{M,S}\;\chi_{M,S} + \phi_{M,A}\;\chi_{M,A}\right] \otimes |\{3^*\}\rangle_c.$$

Therefore, the conclusion is: **The (hard) gluon corrections do not lead to isospin-dependence of the BBM-couplings.**

C. SU(6)-Breaking, Di-quarks, and Isospin Dependence QPC Meson-couplings?

It is well known that $SU(6)_\sigma$ is not a good dynamical symmetry. $SU(6)_W$ is better symmetry but also broken on the $^3P_0$-model. This may be an alternative to solve the "$I_M=1$-problem". Here, we study SU(6)-breaking by describing the $J^P=(1/2)^+$ baryon-states in the $\{56\}$-irrep the flavor-spin-color states by introducing di-quark correlations. These are introduced by the generalization of the SU(6) representation of the baryon states

$$|(8,\mathbf{2})\rangle = \left[w_S\;\phi_{M,S}\;\chi_{M,S} + w_A\;\phi_{M,A}\;\chi_{M,A}\right] \otimes |\{3^*\}\rangle_c, \tag{15}$$

where $w_S^2+w_A^2=1$ and $w_S=w_A=1/\sqrt{2}$. Allowing w_S and w_A to deviate from the SU(6)-weights would break SU(6)-symmetry, and the ratio of the $I_M=0$- and $I_M=1$-coupling is

changes compared to the symmetric case. Effectively, this may be desribed phenomenological by making the pair-creation constant $\gamma_{q\bar{q}}$ isospin dependent. However, the problem with this proposal is *statistics*. For $w_S \neq w_A$ the state is nor symmetric nor antisymmetric. Hence the total 3-quark state does not obey FD-statistics!
Fortunately, effectively the same result can be realized in accordance with FD-statistics by using the baryon-states [25, 26]

$$\begin{aligned}
|(8,\mathbf{2})\rangle_S \sim & \; N_S \Big[f_{12}\ \phi_{M,S}(1,2)\chi_{M,S}(1,2) + f_{13}\ \phi_{M,S}(1,3)\chi_{M,S}(1,3) + \\
& + f_{23}\ \phi_{M,S}(2,3)\chi_{M,S}(2,3) \Big], \\
|(8,\mathbf{2})\rangle_A \sim & \; N_A \Big[g_{12}\ \phi_{M,A}(1,2)\chi_{M,A}(1,2) + g_{13}\ \phi_{M,A}(1,3)\chi_{M,A}(1,3) + \\
& + g_{23}\ \phi_{M,A}(2,3)\chi_{M,A}(2,3) \Big], \qquad (16)
\end{aligned}$$

where $N_{S,A}$ are normalization constants, $f_{12} \equiv |f_1(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3)\rangle$ etc., and similarly for the g_{ij}-functions. Then, the physical baryon states are

$$|(8,\mathbf{2})\rangle = w_S\ |(8,\mathbf{2})\rangle_S + w_A\ |(8,\mathbf{2})\rangle_A, \qquad (17)$$

with $w_S^2 + w_A^2 = 1$. We omitted here the color-component which is identical to that for the SU(6)-symmetric state.
In (16) f_{ij} and g_{ij} are symmetric respectively antisymmetric in the quarks i and j.
In our computations each term with f_{ij} or g_{ij} gives identical results and can be added. This gives the factor 3 from the summation over the quarks of the baryon. In effect, the results for $f_{ij} \neq g_{ij}$ are similar to those with the "wrong" representation of the baryon-state (15).

In Table VIII we show an example where SU(6)-breaking has been applied. Since there is a distiction of the S-type and the A-type quark-pairs, there is also a distiction between $i_{ij} = 0$ and 1, and ipso facto a diference between the $I_M = 0$ and $I_M = 1$ coupling. We have modified only the strength of the pair-creation process for $I_M = 1$. In principle there is also a corresponding change for $I_M = 0$, of course. In Table VIII the SU(6)-breaking is described by the replacement $\gamma_M \to \bar{\gamma}_M$ for the $I_M = 1$ mesons. We do not include this until we have some concrete calculated estimate of these shifts.

D. SU(6)-Breaking, ($\underline{56}$)- and ($\underline{70}$)-irrep Mixing

It is well known that SU$(6)_\sigma$ is not a good dynamical symmetry. SU$(6)_W$ is better symmetry but also broken on the $^3P_0$-model. This may be an alternative to solve the "I_M = 1-problem". Here, we study SU(6)-breaking by describing the $J^P = (1/2)^+$ baryon-states in the {56}-irrep the flavor-spin states by introducing mixng with the {70}-irrep.
From [27] we describe the mixing for the SU3-octet baryons by

$$|\{8\}, \frac{1}{2}^+\rangle = \cos\varphi\ |(\underline{56}, L=0^+)_{N=0}, P_{11}\rangle + \sin\varphi\ |(\underline{70}, L=0^+)_{N=1}, P_{11}\rangle. \qquad (18)$$

In [27] it was found that $\varphi \approx -22^o$. The ($\underline{56}$) wave functions are [24]

$$|(\underline{56}, L=0^+)_{N=0}, P_{11}\rangle = \frac{1}{\sqrt{2}} \left[\chi_{M,A}\ \phi_{M,A} + \chi_{M,S}\ \phi_{M,S}\right]\ \psi^S \qquad (19)$$

TABLE VIII: SU(6)-breaking in ESC08c Couplings. $^3P_0$- and $^3S_1$-model relations, with ideal mixing for vector and scalar meson nonets. Pseudoscalar- and axial-nonets the mixing angles are -23^0 and -42.7^o respectively. We defined $\widetilde{X}_M$, i.e. the isospin factor. Here, Λ_{QPC} = 350 MeV, $\gamma(\alpha_s = 0.30) = 2.19$ etc. The weights are A=1.287 and B=0.738 for the $^3P_0$ and $^3S_1$ respectively.

Meson	$r_M[fm]$	$\widetilde{X}_M$	γ_M	$\bar{\gamma_M}/\gamma_M$	$^3S_1$	$^3P_0$	QPC	ESC08c
$\pi(140)$	0.36	2	6.89	0.8	$g = -4.29$	$g = 7.48$	3.19	3.63
$\eta'(957)$	0.90	6	2.22	1.0	$g = -2.71$	$g = 4.73$	2.02	2.28
$\rho(770)$	0.71	1	2.37	0.8	$g = -0.37$	$g = 1.29$	0.92	0.69
$\omega(783)$	0.71	3	2.35	1.0	$g = -1.40$	$g = 4.78$	3.38	3.52
$a_0(962)$	0.90	$3\sqrt{2}$	2.22	0.8	$g = 0.27$	$g = 0.47$	0.74	0.87
$\epsilon(760)$	0.71	$9\sqrt{2}$	2.37	1.0	$g = 1.74$	$g = 3.04$	4.78	4.74
$a_1(1270)$	0.66	$2\sqrt{2}$	2.09	0.8	$g = -0.21$	$g = -0.74$	-0.95	-1.11
$d_1(1280)$	0.90	$6\sqrt{2}$	2.09	1.0	$g = -0.39$	$g = -1.34$	-1.73	-1.06
$b_1(1235)$	0.90		2.09	0.8	$f = 0$	$f = 0$	0.0	-0.20

$$\begin{aligned} |(\underline{70}, L=0^+)_{N=1}, P_{11}\rangle &= \frac{1}{2}\left[(\psi_{M,S}\chi_{M,A} + \psi_{M,A}\chi_{M,S})\ \phi_{M,A} + \right. \\ &\quad \left.(\psi_{M,A}\chi_{M,A} - \psi_{M,S}\chi_{M,S})\ \phi_{M,S}\right] \\ &\Rightarrow \frac{1}{2}\left[\chi_{M,A}\ \phi_{M,A} - \chi_{M,S}\ \phi_{M,S}\right]\ \psi_{M,S}. \end{aligned} \tag{20}$$

Here, we have choosen to symmetrize (M,S) or anti-symmetrize (M,A) w.r.t. the quarks numbered 1 and 2. The notation in [27] is $\chi' = \chi_{M,A}$, and $\chi'' = \chi_{M,S}$, and similarly for ϕ and ψ. The wave function ψ^S has the complete symmetry w.r.t. 1,2,3 permutations (S_3-group). The last line applies because the overlap integrals in momentum-space vanishes for $\psi_{M,A}$ and ψ^S, see Appendix C. Rewriting, we have

$$\begin{aligned} |(\underline{70}, L=0^+)_{N=1}, P_{11}\rangle &= \frac{1}{2}\left[(\chi_{M,A}\ \phi_{M,A} - \chi_{M,S}\ \phi_{M,S})\ \psi_{M,S}\right. \\ &\quad \left.(\chi_{M,A}\ \phi_{M,S} + \chi_{M,S}\ \phi_{M,A})\ \psi_{M,A}\right], \end{aligned} \tag{21}$$

and, in general, the interference between the irreps is given by

$$\begin{aligned}
&\langle(\underline{70}, L=0^+)_{N=1}, P_{11}|(\underline{56}, L=0^+)_{N=0}, P_{11}\rangle \Rightarrow \\
&\frac{1}{2\sqrt{2}}\left(\chi_{M,A}\ \phi_{M,A} - \chi_{M,S}\ \phi_{M,S}|O^{(1)}_{F,S}|\chi_{M,A}\ \phi_{M,A} + \chi_{M,S}\ \phi_{M,S}\right)\ \left(\psi_{M,S}|O^{(1)}_{mom.sp.}|\psi^S\right) + \\
&\frac{1}{2\sqrt{2}}\left(\chi_{M,S}\ \phi_{M,A} + \chi_{M,A}\ \phi_{M,S}|O^{(2)}_{F,S}|\chi_{M,A}\ \phi_{M,A} + \chi_{M,S}\ \phi_{M,S}\right)\ \left(\psi_{M,A}|O^{(2)}_{mom.sp.}|\psi^S\right).
\end{aligned} \tag{22}$$

The next task is to evaluate the matrix elements of the flavor (isospin) and spin operators on the quark level.

1. Isospin operators: We evaluate below the matrix elements

$$\left(\underline{56}\middle|\sum_{i=1,2,3}\boldsymbol{\tau}^{(i)}\middle|\underline{56}\right)\ ,\ \left(\underline{70}\middle|\sum_{i=1,2,3}\boldsymbol{\tau}^{(i)}\middle|\underline{56}\right).$$

Because of the complete symmetry, we choose quark 3, and consider the matrix elements of τ_z for the proton (P) and the neutron (N). We get

$$\begin{aligned}
&\left(\phi_{M,S}|\tau_z^{(3)}|\phi_{M,S}\right) \Rightarrow \\
&P\ :\ \frac{1}{6}\left(UDU + DUU - 2UUD|\tau_z^{(3)}|UDU + DUU - 2UUD\right) = \frac{1}{6}(+1+1-4) = -\frac{1}{3}, \\
&N\ :\ \frac{1}{6}\left(UDD + DUD - 2DDU|\tau_z^{(3)}|UDD + DUD - 2DDU\right) = \frac{1}{6}(-1-1+4) = +\frac{1}{3},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\phi_{M,A}|\tau_z^{(3)}|\phi_{M,A}\right) \Rightarrow \\
&P\ :\ \frac{1}{2}\left(UDU - DUU|\tau_z^{(3)}|UDU - DUU\right) = \frac{1}{2}(+1+1) = +1, \\
&N\ :\ \frac{1}{2}\left(UDD - DUD|\tau_z^{(3)}|UDD - DUD\right) = \frac{1}{2}(-1-1) = -1,
\end{aligned}$$

which gives on the baryon level

$$\begin{aligned}
\left(\underline{56}|\tau_z^{(3)}|\underline{56}\right) &= \frac{1}{2}\left(\phi_{M,A} + \phi_{M,S}|\tau_z^{(3)}|\phi_{M,A} + \phi_{M,S}\right) \to \frac{1}{3}\tau_z, \\
\left(\underline{70}|\tau_z^{(3)}|\underline{56}\right) &= \frac{1}{2\sqrt{2}}\left(\phi_{M,A} - \phi_{M,S}|\tau_z^{(3)}|\phi_{M,A} + \phi_{M,S}\right) \to \frac{2\sqrt{2}}{3}\tau_z.
\end{aligned} \tag{23}$$

2. Spin operators: This completely analogous to the isospin operators, and we have on the baryon level

$$\begin{aligned}
\left(\underline{56}|\sigma_z^{(3)}|\underline{56}\right) &= \frac{1}{2}\left(\phi_{M,A} + \phi_{M,S}|\sigma_z^{(3)}|\phi_{M,A} + \phi_{M,S}\right) \to \frac{1}{3}\sigma_z, \\
\left(\underline{70}|\sigma_z^{(3)}|\underline{56}\right) &= \frac{1}{2\sqrt{2}}\left(\phi_{M,A} - \phi_{M,S}|\sigma_z^{(3)}|\phi_{M,A} + \phi_{M,S}\right) \to \frac{2\sqrt{2}}{3}\sigma_z.
\end{aligned} \tag{24}$$

3. Isospin-spin operators: We evaluate below the matrix elements

$$\left(\underline{56}\left|\sum_{i=1,2,3}\boldsymbol{\tau}^{(i)}\boldsymbol{\sigma}^{(i)}\right|\underline{56}\right)\ ,\ \left(\underline{70}\left|\sum_{i=1,2,3}\boldsymbol{\tau}^{(i)}\boldsymbol{\sigma}^{(i)}\right|\underline{56}\right).$$

Taking again quark 3 for the proton (P) and the neutron (N), we get now

$$\begin{aligned}
&\Big(\phi_{M,S}|\tau_z^{(3)}\sigma_z^{(3)}|\phi_{M,S}\Big)\Rightarrow\\
&(P,s=\pm1/2)\ :\ \frac{1}{6}\Big(UDU+DUU-2UUD|\tau_z^{(3)}|UDU+DUU-2UUD\Big)=\pm\frac{1}{9},\\
&(N,s=\pm1/2)\ :\ \frac{1}{6}\Big(UDD+DUD-2DDU|\tau_z^{(3)}|UDD+DUD-2DDU\Big)=\mp\frac{1}{9},
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\phi_{M,A}|\tau_z^{(3)}\sigma_z^{(3)}|\phi_{M,A}\Big)\Rightarrow\\
&(P,s=\pm1/2)\ :\ \frac{1}{2}\Big(UDU-DUU|\tau_z^{(3)}|UDU-DUU\Big)=\pm1,\\
&(N,s=\pm1/2)\ :\ \frac{1}{2}\Big(UDD-DUD|\tau_z^{(3)}|UDD-DUD\Big)=\mp1,
\end{aligned}$$

which gives on the baryon level

$$\begin{aligned}
\Big(\underline{56}|\tau_z^{(3)}\sigma_z^{(3)}|\underline{56}\Big)&=\frac{1}{2}\left(\phi_{M,A}+\phi_{M,S}|\tau_z^{(3)}\sigma_z^{(3)}|\phi_{M,A}+\phi_{M,S}\right)\to\frac{5}{9}\tau_z\sigma_z,\\
\Big(\underline{70}|\tau_z^{(3)}\sigma_z^{(3)}|\underline{56}\Big)&=\frac{1}{2\sqrt{2}}\left(\phi_{M,A}-\phi_{M,S}|\tau_z^{(3)}\sigma_z^{(3)}|\phi_{M,A}+\phi_{M,S}\right)\to\frac{2\sqrt{2}}{9}\tau_z\sigma_z.
\end{aligned}\tag{25}$$

Application to the different meson types the ($\underline{56}$) → ($\underline{70}$) transition matrix elements are:

1. Pseudoscalar mesons: baryon-quark matrix elements

$$(\chi^\dagger\boldsymbol{\sigma}\chi_\alpha)\ ,\ (\phi^\dagger\{1,\boldsymbol{\tau}\}\phi_\alpha)\ \to\Big(2\sqrt{2},\frac{2}{3}\sqrt{2}\Big)\,,\tag{26}$$

2. Vector mesons: baryon-quark matrix elements

$$(\chi^\dagger 1\ \chi_\alpha)\ ,\ (\phi^\dagger\{1,\boldsymbol{\tau}\}\phi_\alpha)\ \to\Big(0,2\sqrt{2}\Big)\,,\tag{27}$$

3. Scalar mesons: baryon-quark matrix elements

$$(\chi^\dagger 1\ \chi_\alpha)\ ,\ (\phi^\dagger\{1,\boldsymbol{\tau}\}\phi_\alpha)\ \to\Big(0,2\sqrt{2}\Big)\,,\tag{28}$$

4. Axial-vector mesons: baryon-quark matrix elements

$$(\chi^\dagger\boldsymbol{\sigma}\chi_\alpha)\ ,\ (\phi^\dagger\{1,\boldsymbol{\tau}\}\phi_\alpha)\ \to\Big(2\sqrt{2},\frac{2}{3}\sqrt{2}\Big)\,.\tag{29}$$

In order to have a non-zero transition between the SU(6)-irreps, one needs at least one non-trivial operator. Therefore, in the case of the I=0 vector- (ω) and scalar-mesons (ϵ) there is no effect of SU(6)-breaking via SU(6)-irrep mixing.

Inclusion of the quark-gluon form factor and SU(6)-breaking via irrep-mixing, we obtain, using results from Appendices B-D,

$$g_V(I=1) = g_V(I=1)\ F^{(L=0)}_{QQG}(R_A, R_M, \Lambda_{QQG})\ \Big(1+\sin(2\varphi)\ \langle\tau\rangle\ I^{L=0)\prime\prime}(0)/I^{(L=0)}(0)\Big)\,,$$
$$g_S(I=1) = g_S(I=1)\ F^{(L=1)}_{QQG}(R_A, R_M, \Lambda_{QQG})\ \Big(1+\sin(2\varphi)\ \langle\tau\rangle\ I^{L=1)\prime\prime}(0)/I^{(L=1)}(0)\Big)\,. \tag{30}$$

Similarly for g_P and g_A. In Table VI the results are shown for $\varphi = -22^o$ [27]. We note that the effects for the L=1 mesons is small, because of the large Λ_{QQG}. The values in parentheses are with $\varphi = 0$, i.e. no SU(6)-breaking included. The results show a dominance for the $^3P_0$-over the $^3S_1$-mechanism with a ratio 2:1.
Now it is appropriate to scale A, B, and Λ_{QCD} such that A+B=1. Equivalent to the set $\{A, B, \Lambda_{QCD}\} = \{1.445, 0.772, 350\}$ is the set $\left\{\bar{A}, \bar{B}, \bar{\Lambda}_{QCD}\right\} = \{0.652, 0.348, 235\}$. The last set is the proper one because then the given γ values are genuine.

E. Hard-gluon corrections to BBM-couplings

Here, we compute the so-called rescaled quark operators due to the hard-gluon renormalization of the BBM-vertex, see the diagram on the r.h.s. of Fig. 4. The operator $\otimes$ stands for the $(\bar{q}_s q_s)(\bar{q}_j \Gamma\ q_i)$ operator, where s denotes the 'spectator' quark and $\Gamma = 1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu$. The latter indicate the type of meson that couples in the vertex. The diagram has the value [28]

$$\begin{aligned} G_{qqX} &= \int \frac{d^4k}{(2\pi)^4}(-ig)^2\ \left(\frac{-i}{k^2}\right)\left(\bar{q}_s\lambda^a\gamma^\mu \frac{i(\not{k}_2+\not{k}+m)}{(k_2+k)^2-m^2}\lambda^a\gamma^\mu q_s\right)\cdot \\ &\qquad \times\left(\bar{q}_j\frac{i(\not{k}_3-\not{k}+m)}{(k_3-k)^2-m^2}\Gamma\ \lambda^a\gamma_\mu\right) \\ &\sim +ig^2\int\frac{d^4k}{(2\pi)^4}\ \frac{k^2/d}{(k^2)^3}\,(\bar{q}_s\gamma^\rho\gamma^\mu\lambda^a\ q_s)\ \ (\bar{q}_j\gamma_\rho\ \Gamma\ \gamma_\mu\lambda^a q_i) \\ &= -\frac{1}{4}g^2\ \frac{\Gamma(2-d/2)}{(4\pi)^2}\,(\bar{q}_s\gamma^\rho\gamma^\mu\lambda^a\ q_s)\ \ (\bar{q}_j\gamma_\rho\ \Gamma\ \gamma_\mu\lambda^a q_i)\,. \end{aligned}$$

Here, $m = m_Q$ and $\sim$ means the extraction of the divergent part of the integral. The symmetry of the k-integral has been used to deal with the $\not{k}$ operators.
In principle the gluon-corrections are finite, because we have in addition an extra (confined) gluon/scalar-propagator in the $\otimes$-vertex. So, we assume that the approximate computation above is only valid in the region of integration where $k^2 \leq \Lambda$. In a direct QCD-calculation the ultraviolet-divergent terms in the evaluation of Fig. 4 would be replaced by logarithms

TABLE IX: SU(6)-breaking in coupling constants, using (56) and (70)-irrep mixing with angle $\varphi = -22^o$ for the $^3P_0$- and $^3S_1$-model. Gaussian Quark-gluon cut-off $\Lambda_{QQG} = 986.6$ MeV. Ideal mixing for vector and scalar meson nonets. Pseudoscalar- and axial-nonets the mixing angles are -23^0 and -42.7^o respectively, imposing the OZI-rule. Here, $\Lambda_{QPC} = 235$ MeV, $\gamma(\alpha_s = 0.30) = 2.19$ etc. The weights are A=0.652 and B=0.348 for the $^3P_0$ and $^3S_1$ respectively.

Meson	$r_M[fm]$	γ_M	$^3S_1$	$^3P_0$	QPC	ESC08c
$\pi(140)$	0.23	5.51	$g = -4.11$	$g = +7.70$	3.58 (3.88)	3.63
$\eta'(957)$	0.71	2.22	$g = -2.60$	$g = +4.87$	2.27 (2.72)	2.28
$\rho(770)$	0.71	2.37	$g = -0.28$	$g = +1.03$	0.75 (0.93)	0.69
$\omega(783)$	0.71	2.35	$g = -1.28$	$g = +4.79$	3.51 (3.51)	3.52
$a_0(962)$	0.81	2.22	$g = +0.32$	$g = +0.61$	0.93 (0.97)	0.87
$\epsilon(760)$	0.71	2.37	$g = +1.65$	$g = +3.08$	4.73 (4.73)	4.74
$a_1(1270)$	0.61	2.09	$g = -0.24$	$g = -0.88$	-1.13 (-1.13)	-1.11
$e_1(1420)$	0.61	2.09	$g = -0.23$	$g = -0.85$	-1.08 (-1.10)	-1.06

cut off at Λ [28]. The lower limit we take as m_Q. Thus the correction should be evaluated by replacing

$$g^2 \frac{\Gamma(2-d/2)}{(4\pi)^2} \rightarrow \frac{\alpha_s}{4\pi}\ \log\left(\frac{\Lambda^2}{m_Q^2}\right).$$

with this interpretation the hard-gluon correction is given by

$$(\bar{q}_s\ q_s)\ (\bar{q}_j\Gamma\ q_i)\ \rightarrow\ \left(1+\frac{\alpha_s}{4\pi}\ \log\left(\frac{\Lambda^2}{m_Q^2}\right)\right)(\bar{q}_s\ q_s)\ (\bar{q}_j\Gamma\ q_i).$$

F. Meson-radii and CS-Goldstone-boson exchange

Since the constituent quarks have a significant mass ≈ 330 MeV, the exchange of the Chiral-Symmetry Goldstone-bosons (GBE) may be operating between quarks [29]. For π-

exchange in a $q\bar{q}$-sytem gives, see [30],

$$V_{q\bar{q}}(\pi) \;=\; +\frac{g^2_{\pi qq}}{4\pi}\frac{m^3_\pi}{12m^2_Q}\,(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\;(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\;\phi^1_C(r),$$

which has the same sign as for NN because the C-parity of the π^0 is +1, see [31]. The pion-nucleon form factor mass $\Lambda_{\pi NN} \approx 900$ MeV, which is larger than what would be expected from the quark model and the radius of the nucleon. Therefore, a gaussian form factor for the πqq-coupling must be rather large, and the π-exchange between quarks in a baryon is dominated by the tail. *This is opposite to the GBE-asumption in [29], where it is assumed that GBE is dominated by the cut-off part.*' Now, for the spin-triplet $q\bar{q}$-states this potential is repulsive for $I_M = 1$ and atractive for $I_M = 0$. This applies to the vector-, scalar-, and axial-vector mesons. Then, for these mesons one expects larger radii for the $I_M = 1$- than for the $I_M = 0$-mesons. In the case of the pseudoscalar mesons the opposite would be true. Except for the axial-vector-mesons, these features are confirmed in Table's IV and V.
Note: In the argument above we did not include the possible effect of the tensor and spin-orbit force for the P-wave $q\bar{q}$-states. ($^3P_0 : S_{12} = -4, L.S = -2$, $^3P_1 : S_{12} = +2, L.S = -1$.)

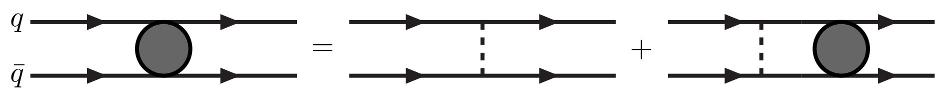

FIG. 5: *OBE and BSE in $q\bar{q}$-systems*

G. Meson-radii and OBE-exchange

Here we analyze the expectation w.r.t. to the quark radii in mesons, where we suppose that OBE-exchange in the $q\bar{q}$-systems is important. In Fig. 5 we envisage the $q\bar{q}$ wave function generated through solving the Bethe-Salpeter equation (BSE) with OBE as the driving force. The meson-echange central, spin-spin, and tensor potentals are [30]

$$\begin{aligned}
V_C &= -\frac{m_V}{4\pi}\,g^2_V\,\phi^0_{C,V} - \frac{m_S}{4\pi}\,g^2_S\,\phi^0_{C,S},\\
V_\sigma &= +\frac{m_P}{4\pi}\,g^2_P\,\frac{m^2_P}{12M^2_Q}\,\phi^1_{C,P} - \frac{m_V}{4\pi}\,(g_V+f_V)^2\,\frac{m^2_V}{6M^2_Q}\,\phi^1_{C,V} + \frac{m_A}{4\pi}\,g^2_A\,\phi^0_{C,A},\\
V_T &= +\frac{m_P}{4\pi}\,g^2_P\,\frac{m^2_P}{4M^2_Q}\,\phi^0_{T,P} + \frac{m_V}{4\pi}\,(g_V+f_V)^2\,\frac{m^2_V}{4M^2_Q}\,\phi^0_{T,V} - \frac{m_A}{4\pi}\,g^2_A\,\frac{m^2_A}{4M^2_Q}\,\phi^0_{T,A}.
\end{aligned}$$

There are also central potentials from Pomeron and Odderon exchange, but they have no isospin dependence.
As for the isospin dependence, neglecting mass differences one has $g^2_X \sim g^2_{X,0} + \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,g^2_{X,1}$, **which can be expected all to be positive**. We notice the strong central attraction,

which is stronger for $I_M = 1$ than for $I_M = 0$. *Assuming this to be dominant, we expect that ipso facto $R_M(I = 1) < R_M(I = 0)$, contrary to what we seem to find from the couplings obtained in ESC08c!*

H. QCD-Sum Rules

To make connection with the QCD-SR approach, we have using ρ-saturation

$$Im\ \Pi(s) = \frac{\pi m_\rho^2}{f_\rho^2}\ \delta(s - m_\rho^2) \tag{31}$$

and [32], eqn. (4.44)

$$\int e^{-s/M^2} Im\Pi(s)\ ds = \frac{1}{8\pi} M^2 \left[1 + \frac{\alpha_s}{\pi}\right]\ , \tag{32}$$

which yields

$$\frac{\pi m_\rho^2}{f_\rho^2} e^{-m_\rho^2/M^2} = \frac{1}{8\pi} M^2 \left[1 + \frac{\alpha_s}{\pi}\right]\ . \tag{33}$$

With $M = m_\rho$ one finds

$$\frac{f_\rho^2}{4\pi} = \frac{2\pi}{e}\left(1 + \frac{\alpha_s}{\pi}\right)^{-1} \approx \frac{2\pi}{e} = 2.33\ . \tag{34}$$

The basis for this determination are the simplest vacuum-polarization graphs, the 1 corresponds the 1-loop quarkdiagram, and the α_s term to the one-gluon exchange correction to that diagram.
The connection with the pair creation constant γ is given by for example (5) and (10). One has

$$\begin{aligned} \frac{\gamma^2}{4\pi} &= \frac{f_\rho^2}{4\pi} \cdot \frac{3\pi}{4} \cdot \frac{m_\pi^2}{m_\rho^2} \\ &= \frac{3\pi^2}{2e}\left(1 + \frac{\alpha_s}{\pi}\right)^{-1}\ \frac{m_\pi^2}{m_\rho^2}\ . \end{aligned} \tag{35}$$

The 'universality' of γ is explained by $m_\rho \approx m_\epsilon \approx m_\omega$, and $f_\rho \approx f_\omega \approx f_\epsilon$, so that instead of ρ-like quantities in (11) one could as well use ϵ-like quantities, etc.

What remains puzzling is why the QCD-SR approach does not give, like $^3P_0$-model, the ratio's $(I = 0) : (I = 1) \sim 3 : 1$?!!

APPENDIX A: HARMONIC-OSCILLATOR MOMENTUM SPACE WAVE-FUNCTIONS

The harmonic-oscillator wave functions in configuration space is [33]

$$\psi_{n,l,m}(\mathbf{r}) \;=\; N_{n,l}\exp\left(-\frac{1}{2}\lambda r^2\right)\;\left(\sqrt{\lambda}r\right)^l\;L_k^{l+1/2}(\lambda r^2)\;Y_m^l(\theta,\phi),$$
$$n = l+2k\;(k=0,1,2,\ldots)\;\;,\;\;\lambda=\frac{m\omega}{\hbar}\equiv R_M^{-2}\;. \tag{A1}$$

Here, $L_k^{l+1/2}(t)$ are the Laguerre polynomials, and $k=n_r$ as used in some of the literature. In Table I the for this paper relevant entities are given. The Fourier transformation to

TABLE I: H.O. Laguerre polynomials and Normalizations

n=0, l=0, k=0	$L_0^{1/2}=1$	$N_{0,0}=2\left(\frac{\lambda^3}{\pi}\right)^{1/4}$
n=2, l=0, k=1	$L_1^{1/2}=\frac{3}{2}-\lambda r^2$	$N_{2,0}=2\sqrt{\frac{2}{3}}\left(\frac{\lambda^3}{\pi}\right)^{1/4}$
n=1, l=1, k=0	$L_1^{3/2}=1$	$N_{1,1}=2\sqrt{\frac{2}{3}}\left(\frac{\lambda^3}{\pi}\right)^{1/4}$
n=3, l=1, k=1	$L_1^{3/2}=\frac{5}{2}-\lambda r^2$	$N_{3,1}=2\sqrt{\frac{4}{15}}\left(\frac{\lambda^3}{\pi}\right)^{1/4}$

momentum-space is given by

$$\widetilde{\psi}_{n,l,m}(\mathbf{k}) \;=\; (2\pi)^{-3/2}\int d^3r\;\psi_{n,l,m}(\mathbf{r})\;\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)\;, \tag{A2}$$

where $\mathbf{k}$ is the relative momentum in the $Q\bar{Q}$ CM-system. So, denoting the quark and anti-quark CM-momenta by $\mathbf{k}_1$ and $\mathbf{k}_2$, we have $\mathbf{k}=(\mathbf{k}_1-\mathbf{k}_2)/2$. Then, for $\widetilde{\psi}_{n,l,m}(\mathbf{k}_1,\mathbf{k}_2)\equiv\widetilde{\psi}_{n,l,m}(\mathbf{k})$ we have the normalization

$$\int d^3k_1d^3k_2\;\delta(\mathbf{k}_1+\mathbf{k}_2)\;|\widetilde{\psi}_{n,l,m}(\mathbf{k}_1,\mathbf{k}_2)|^2=1\;. \tag{A3}$$

1. Case $n=0,l=0$:

$$\begin{aligned}\widetilde{\psi}_{0,0,0}(\mathbf{k}) \;&=\; (2\pi)^{-3/2}\left(\frac{\lambda}{\pi}\right)^{3/4}\int d^3r\;\exp\left[-\frac{1}{2}\lambda r^2+i\mathbf{k}\cdot\mathbf{r}\right]\\ &=\;(\pi\lambda)^{-3/4}\exp\left(-\mathbf{k}^2/2\lambda\right)\\ &=\;\left(\frac{R_M^2}{\pi}\right)^{3/4}\exp\left[-\frac{R_M^2}{8}\left(\mathbf{k}_1-\mathbf{k}_2\right)^2\right]\;.\end{aligned} \tag{A4}$$

Notice that we included here in the first step the factor $Y_0^0(\theta,\phi) = 1/\sqrt{4\pi}$ in $\psi_{0,0,0}$.

2. Case $n = 2, l = 0$: Similarly to the previous case we now find

$$\begin{aligned}
\widetilde{\psi}_{2,0,0}(\mathbf{k}) &= (2\pi)^{-3/2}\sqrt{\frac{2}{3}}\left(\frac{\lambda}{\pi}\right)^{3/4}\int d^3r\ \left(\frac{3}{2}-\lambda r^2\right)\exp\left[-\frac{1}{2}\lambda r^2 + i\mathbf{k}\cdot\mathbf{r}\right] \\
&= -\sqrt{\frac{3}{2}}\,(\pi\lambda)^{-3/4}\left[1-\frac{2}{3}\mathbf{k}^2/\lambda\right]\exp\left(-\mathbf{k}^2/2\lambda\right) \\
&= -\sqrt{\frac{3}{2}}\left(\frac{R_M^2}{\pi}\right)^{3/4}\left\{1-\frac{1}{6}\left(\mathbf{k}_1-\mathbf{k}_2\right)^2 R_M^2\right\}\cdot \\
&\quad\times\exp\left[-\frac{R_M^2}{8}\left(\mathbf{k}_1-\mathbf{k}_2\right)^2\right]\ . \qquad \text{(A5)}
\end{aligned}$$

3. Case $n = 1, l = 1$: For $l = 1$ it is convenient to use cartesian components of the wave-function by the substitution $Y_m^1(\theta,\phi) \to \sqrt{3/4\pi}r_i, i = 1,2,3$. Then, we find for the wave-function in momentum space

$$\begin{aligned}
\widetilde{\psi}_{1,1,i}(\mathbf{k}) &= (2\pi)^{-3/2}\sqrt{\frac{2}{\pi}}\left(\frac{\lambda^3}{\pi}\right)^{1/4}\int d^3r\ \left(\sqrt{\lambda}\ r_i\right)\ \exp\left[-\frac{1}{2}\lambda r^2 + i\mathbf{k}\cdot\mathbf{r}\right] \\
&= (2\pi)^{-3/2}\sqrt{\frac{2}{\pi}}\left(\frac{\lambda^5}{\pi}\right)^{1/4}(-i\nabla_{k,i})\int d^3r\ \exp\left[-\frac{1}{2}\lambda r^2 + i\mathbf{k}\cdot\mathbf{r}\right] \\
&= i\sqrt{\frac{2}{\pi}}\left(\pi\lambda^5\right)^{-1/4}\exp\left(-\mathbf{k}^2/2\lambda\right)\cdot k_i\ . \qquad \text{(A6)}
\end{aligned}$$

4. Case $n = 3, l = 1$: Similarly to the former case we now find

$$\begin{aligned}
\widetilde{\psi}_{3,1,i}(\mathbf{k}) &= (2\pi)^{-3/2}\sqrt{\frac{4}{5\pi}}\left(\frac{\lambda^3}{\pi}\right)^{1/4}\int d^3r\ \left(\sqrt{\lambda}\ r_i\right)\left(\frac{5}{2}-\lambda r^2\right)\exp\left[-\frac{1}{2}\lambda r^2 + i\mathbf{k}\cdot\mathbf{r}\right] \\
&\Rightarrow -i\sqrt{\frac{5}{\pi}}\left(\pi\lambda^5\right)^{-1/4}\left(1-\frac{2}{5}\mathbf{k}^2/\lambda\right)\exp\left(-\mathbf{k}^2/2\lambda\right)\cdot k_i\ . \qquad \text{(A7)}
\end{aligned}$$

Also the cases 4 and 5 can be transcribed to the spherical base in an obvious manner.

Summarizing, we have the following momentum-space $Q\bar{Q}$ h.o. wave functions:

$$(n=0,l=0)\ :\ \widetilde{\psi}_{0,0,0}(\mathbf{k}) = +\left(\frac{R_M^2}{\pi}\right)^{3/4}\exp\left[-\frac{R_M^2}{8}\left(\mathbf{k}_1-\mathbf{k}_2\right)^2\right]\ , \qquad \text{(A8)}$$

$$\begin{aligned}
(n=2,l=0)\ :\ \widetilde{\psi}_{2,0,0}(\mathbf{k}) &= -\sqrt{\frac{3}{2}}\left(\frac{R_M^2}{\pi}\right)^{3/4}\left\{1-\frac{1}{6}\left(\mathbf{k}_1-\mathbf{k}_2\right)^2 R_M^2\right\}\cdot \\
&\quad\times\exp\left[-\frac{R_M^2}{8}\left(\mathbf{k}_1-\mathbf{k}_2\right)^2\right]\ , \qquad \text{(A9)}
\end{aligned}$$

$$(n=1,l=1)\ :\ \widetilde{\psi}_{1,1,m}(\mathbf{k}) = -\frac{1}{\sqrt{2}}\ R_M\ \left(\frac{R_M^2}{\pi}\right)^{3/4}\left[\boldsymbol{\epsilon}_m\cdot\left(\mathbf{k}_1-\mathbf{k}_2\right)\right]\cdot\cdot$$

$$\times \exp\left[-\frac{R_M^2}{8}(\mathbf{k}_1-\mathbf{k}_2)^2\right], \tag{A10}$$

$$(n=3,l=1)\ :\ \widetilde{\psi}_{3,1,m}(\mathbf{k}) = +\frac{1}{2}\sqrt{5\pi}\left(\frac{R_M^2}{\pi}\right)^{5/4}\left\{1-\frac{1}{10}(\mathbf{k}_1-\mathbf{k}_2)^2 R_M^2\right\}\cdot$$
$$\times\left[\boldsymbol{\epsilon}_m\cdot(\mathbf{k}_1-\mathbf{k}_2)\right]\cdot\exp\left[-\frac{R_M^2}{8}(\mathbf{k}_1-\mathbf{k}_2)^2\right]. \tag{A11}$$

Here $\mathcal{Y}_m^1(\theta,\phi)\equiv i|\mathbf{k}|Y_m^1(\theta,\phi)$ [34].

The h.o. energy is given by

$$E_{h.o.} = (n+3/2)\hbar\omega\ . \tag{A12}$$

Then, the energy difference for the radial states is given by

$$\Delta E = \Delta n\hbar\omega = \Delta n\frac{\hbar^2}{mR_M^2} = 2\Delta n\left(\frac{\hbar}{m_F c}/R_M\right)^2\frac{m_F}{m_Q}\cdot m_F c^2\ , \tag{A13}$$

where we used $\lambda = m\omega/\hbar = R_M^{-2}$, so that $\omega=\hbar/(mR_M^2)$, and the reduced mass $m=m_Q/2$. Taking $m_Q = 330$ MeV, and $R_M = 1$ fm, we find for $\Delta n = 2$ that $\Delta E \approx 485$ MeV. For $R_M = 0.66$ fm, $\Delta E \approx 1102$.

In connection with time-reversal invariance it is better to use as basis functions in configuration space the spherical harmonics with an extra i^L [18]

$$\mathcal{Y}_L^m(\theta,\varphi) = i^L\ r^L Y_L^m(\theta,\varphi). \tag{A14}$$

This implies for L=1 an extra factor i also in momentum space. This way the scalar- and axial-vector meson couplings become real.

APPENDIX B: BASIC OVERLAP INTEGRAL QPC-MODEL I

We define the basic overlap integral $I_0(A;B,M)$, where A and B are baryons, as

$$\begin{aligned} I_0(A;B,M) \ \equiv\ & \int d^3k_1\ d^3k_2\ d^3k_3\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot \\ \times\ & \int d^3k_3' d^3k_4 d^3k_5 \delta(\mathbf{k_3}-\mathbf{k_3'}-\mathbf{k_4}-\mathbf{k_5})\ \delta(\mathbf{p'}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot \\ \times\ & \widetilde{\psi}_B^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4)\ \widetilde{\psi}_A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\ \widetilde{\psi}_M^{(L=0)}(\mathbf{k}_3',\mathbf{k}_5)^* \\ =\ & \mathcal{N}_0\int d^3k_1\ d^3k_2\ d^3k_3\ d^3k_4\ d^3k_3'\ \delta(\mathbf{p}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3})\cdot \\ \times\ & \delta(\mathbf{p'}-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_4})\cdot\exp\left[+f(\mathbf{k}_1,\ldots,\mathbf{k}_4;\mathbf{k}_3',\mathbf{k}_5)\right]\ , \end{aligned} \tag{B1}$$

where in the last expression $\mathbf{k}_5$ is understood to be fixed according to the δ-function in the first expression, and

$$\mathcal{N}_0 = \left(\frac{\sqrt{3}R_A^2}{\pi}\right)^3\left(\frac{R_M^2}{\pi}\right)^{3/4}\ , \tag{B2}$$

and

$$
\begin{aligned}
f(\mathbf{k}_1,\ldots,\mathbf{k}_4,\mathbf{k}_3',\mathbf{k}_5) \;=\; & -\frac{1}{6}R_A^2\left[(\mathbf{k}_1-\mathbf{k}_2)^2+(\mathbf{k}_1-\mathbf{k}_3)^2+(\mathbf{k}_2-\mathbf{k}_3)^2\right] \\
& -\frac{1}{6}R_A^2\left[(\mathbf{k}_1-\mathbf{k}_2)^2+(\mathbf{k}_1-\mathbf{k}_4)^2+(\mathbf{k}_2-\mathbf{k}_4)^2\right] \\
& -\frac{1}{8}R_M^2(\mathbf{k}_3'-\mathbf{k}_5)^2 \ . \qquad \text{(B3)}
\end{aligned}
$$

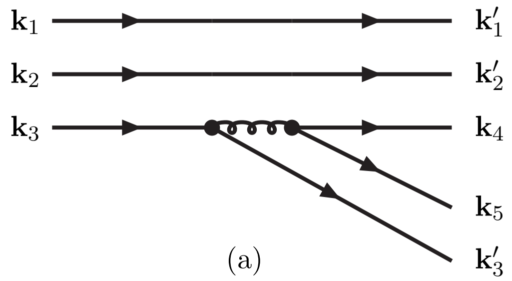

FIG. 1: *Gluon/Scalar-exchange (a).*

Because of the δ-functions, there remain only two three momenta which have to be integrated over in (B1). For these we choose

$$\mathbf{\Delta}_{12}=(\mathbf{k}_1-\mathbf{k}_2) \quad , \quad \mathbf{Q}_{34}=(\mathbf{k}_3+\mathbf{k}_4)/2 \ . \qquad \text{(B4)}$$

From the δ-functions we have

$$\mathbf{p}=(\mathbf{k}_1+\mathbf{k}_2)+\mathbf{k}_3 \quad , \quad \mathbf{p}'=(\mathbf{k}_1+\mathbf{k}_2)+\mathbf{k}_4 \ ,$$

which gives

$$
\begin{aligned}
\mathbf{k}_3-\mathbf{k}_4 &= \mathbf{k}_3'+\mathbf{k}_5=\mathbf{p}-\mathbf{p}'\equiv\mathbf{k} \ , \\
\mathbf{k}_3-\mathbf{k}_3' &= \mathbf{k}_4+\mathbf{k}_5\equiv\mathbf{Q} \ , \\
\mathbf{k}_1+\mathbf{k}_2 &= \frac{1}{2}(\mathbf{p}+\mathbf{p}')-\frac{1}{2}(\mathbf{k}_3+\mathbf{k}_4)\equiv\mathbf{q}-\mathbf{Q}_{34} \ , \qquad \text{(B5)}
\end{aligned}
$$

where $\mathbf{q}=(\mathbf{p}'+\mathbf{p})/2$. From these relations we readily derive that

$$
\begin{aligned}
\mathbf{k}_1 &= \frac{1}{2}\left(\mathbf{q}-\mathbf{Q}_{34}+\mathbf{\Delta}_{12}\right) \ , \\
\mathbf{k}_2 &= \frac{1}{2}\left(\mathbf{q}-\mathbf{Q}_{34}-\mathbf{\Delta}_{12}\right) \ , \\
\mathbf{k}_3 &= \mathbf{Q}_{34}+\frac{1}{2}\mathbf{k} \ , \\
\mathbf{k}_4 &= \mathbf{Q}_{34}-\frac{1}{2}\mathbf{k} \ , \\
\mathbf{k}_3' &= \mathbf{Q}_{34}+\frac{1}{2}\mathbf{k}-\mathbf{Q} \ , \\
\mathbf{k}_5 &= -\mathbf{Q}_{34}+\frac{1}{2}\mathbf{k}+\mathbf{Q} \ . \qquad \text{(B6)}
\end{aligned}
$$

Then,

$$(\mathbf{k}_1-\mathbf{k}_2)^2+(\mathbf{k}_1-\mathbf{k}_3)^2+(\mathbf{k}_2-\mathbf{k}_3)^2=\boldsymbol{\Delta}_{12}^2+$$
$$\frac{1}{4}\left(\mathbf{q}-3\mathbf{Q}_{34}+\boldsymbol{\Delta}_{12}-\mathbf{k}\right)^2+\frac{1}{4}\left(\mathbf{q}-3\mathbf{Q}_{34}-\boldsymbol{\Delta}_{12}-\mathbf{k}\right)^2=$$
$$\boldsymbol{\Delta}_{12}^2+\frac{1}{4}\left(\mathbf{q}-\mathbf{k}\right)^2+\frac{1}{4}\left(\boldsymbol{\Delta}_{12}-3\mathbf{Q}_{34}\right)^2+\frac{1}{2}\left(\mathbf{q}-\mathbf{k}\right)\left(\boldsymbol{\Delta}_{12}-3\mathbf{Q}_{34}\right)$$
$$+\frac{1}{4}\left(\mathbf{q}-\mathbf{k}\right)^2+\frac{1}{4}\left(\boldsymbol{\Delta}_{12}+3\mathbf{Q}_{34}\right)^2-\frac{1}{2}\left(\mathbf{q}-\mathbf{k}\right)\left(\boldsymbol{\Delta}_{12}+3\mathbf{Q}_{34}\right)=$$
$$\frac{3}{2}\boldsymbol{\Delta}_{12}^2+\frac{1}{2}\left(\mathbf{q}-\mathbf{k}\right)^2+\frac{9}{2}\mathbf{Q}_{34}^2-3\left(\mathbf{q}-\mathbf{k}\right)\cdot\mathbf{Q}_{34}$$

Similarly

$$(\mathbf{k}_1-\mathbf{k}_2)^2+(\mathbf{k}_1-\mathbf{k}_4)^2+(\mathbf{k}_2-\mathbf{k}_4)^2=$$
$$\boldsymbol{\Delta}_{12}^2+\frac{1}{4}\left(\mathbf{q}-3\mathbf{Q}_{34}+\boldsymbol{\Delta}_{12}+\mathbf{k}\right)^2+$$
$$\boldsymbol{\Delta}_{12}^2+\frac{1}{4}\left(\mathbf{q}-3\mathbf{Q}_{34}-\boldsymbol{\Delta}_{12}+\mathbf{k}\right)^2\Rightarrow$$
$$\frac{3}{2}\boldsymbol{\Delta}_{12}^2+\frac{1}{2}\left(\mathbf{q}+\mathbf{k}\right)^2+\frac{9}{2}\mathbf{Q}_{34}^2-3\left(\mathbf{q}+\mathbf{k}\right)\cdot\mathbf{Q}_{34}$$

Then,

$$\begin{aligned}f(\mathbf{k}_1,\ldots,\mathbf{k}_4,\mathbf{k}_3',\mathbf{k}_5) &\equiv -\alpha(\mathbf{q}^2+\mathbf{k}^2)+\bar{f}(\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_3',\mathbf{k}_5),\\ \bar{f}(\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_3',\mathbf{k}_5) &= -\alpha\left[3\boldsymbol{\Delta}_{12}^2+9\mathbf{Q}_{34}^2-6\mathbf{q}\cdot\mathbf{Q}_{34}\right]\\ &\quad -4\gamma\left(\mathbf{Q}_{34}-\mathbf{Q}\right)^2,\end{aligned}\tag{B7}$$

where

$$\alpha=\frac{1}{6}R_A^2\quad,\quad\gamma=\frac{1}{8}R_M^2\ .\tag{B8}$$

1. Parametrization Integrand with $\boldsymbol{\Delta}_{12},\mathbf{Q}_{34},\mathbf{Q}$ (B)

Since $\mathbf{k}_1+\mathbf{k}_2=\mathbf{q}-\mathbf{Q}_{34}$, we have

$$\int d^3k_1d^3k_2\Rightarrow\frac{1}{8}\int d^3\Delta_{12}\Rightarrow\frac{1}{8}\left(\frac{\pi}{3\alpha}\right)^{3/2}\ .\tag{B9}$$

Also, because $\mathbf{k}_3-\mathbf{k}_4=\mathbf{k}$ we have $\int d^3k_3d^3k_4\Rightarrow\int d^3Q_{34}$ and

$$\begin{aligned}\bar{I}_0 &= \mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\int d^3Q_{34}\int d^3Q\,\cdot\\ &\quad\times\exp\Big\{-(9\alpha+4\gamma)\mathbf{Q}_{34}^2-4\gamma\mathbf{Q}^2+6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}+8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34}\Big\}\\ &= \mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{4\gamma}\right)^{3/2}\int d^3Q_{34}\ \exp\left\{-9\alpha\mathbf{Q}_{34}^2+6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}\right\}\end{aligned}$$

$$= \mathcal{N}_0 \left(\frac{1}{8}\right) \left(\frac{\pi}{3\alpha}\right)^{3/2} \left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2} \exp\left[\alpha \mathbf{a}^2\right]$$

$$\Rightarrow 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} = 2\sqrt{2}\ \pi^{3/4}\ R_M^{-3/2}, \tag{B10}$$

where $\mathbf{a} \equiv \mathbf{q}$. Above, we performed with $\mathbf{b} \equiv \mathbf{Q}_{34}$ the integral

$$\int d^3Q \exp\Big[-4\gamma\mathbf{Q}^2 + 8\gamma\mathbf{b}\cdot\mathbf{Q}\Big] = \left(\frac{\pi}{4\gamma}\right)^{3/2} \exp\Big(4\gamma\mathbf{b}^2\Big),$$

from which we derive that extra Q-vector components in the integrand lead

$$Q_i \rightarrow (1/8\gamma)\nabla_{b,i} \rightarrow b_i \left(\frac{\pi}{4\gamma}\right)^{3/2} \exp\Big(4\gamma\mathbf{b}^2\Big),$$

$$Q_iQ_j \rightarrow (1/8\gamma)^2\nabla_{b,i}\nabla_{b,j} \rightarrow \left[\frac{1}{8\gamma}\delta_{ij} + b_ib_j\right]\left(\frac{\pi}{4\gamma}\right)^{3/2} \exp\Big(4\gamma\mathbf{b}^2\Big).$$

With extra components of $\mathbf{Q}$ in the integrand, the integrals get the following factors w.r.t. the basic integral $\bar{I}_0$:

$$\begin{aligned}
Q_{34,i} &\rightarrow \frac{1}{6\alpha}\boldsymbol{\nabla}_{a,i} \rightarrow \frac{1}{3}\ a_i, \\
Q_{34,i}Q_{34,j} &\rightarrow \left(\frac{1}{6\alpha}\right)^2 \boldsymbol{\nabla}_{a,i}\boldsymbol{\nabla}_{a,j} \rightarrow \left(\frac{1}{6\alpha}\right)^2 \Big[2\alpha\ \delta_{ij} + 4\alpha^2 a_i\ a_j\Big], \\
Q_{34_i}Q_{34,j}Q_{34,k} &\rightarrow \left(\frac{1}{6\alpha}\right)^3 \boldsymbol{\nabla}_{a,i}\boldsymbol{\nabla}_{a,j}\boldsymbol{\nabla}_{a,k} \rightarrow \left(\frac{1}{6\alpha}\right)^3 \Big[4\alpha^2\Big(\delta_{ij}\ a_k + \delta_{ik}\ a_j + \delta_{jk}\ a_i\Big)\Big], \\
Q_{34_i}Q_{34,j}Q_{34,k}Q_{34,l} &\rightarrow \left(\frac{1}{6\alpha}\right)^4 \boldsymbol{\nabla}_{a,i}\boldsymbol{\nabla}_{a,j}\boldsymbol{\nabla}_{a,k}\boldsymbol{\nabla}_{a,l} \rightarrow \left(\frac{1}{6\alpha}\right)^4 \Big[4\alpha^2\Big(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\Big)\Big], \\
Q_{34,i}Q_j &\rightarrow Q_{34,i}Q_{34,j} \rightarrow \left(\frac{1}{6\alpha}\right)^2 \Big[2\alpha\ \delta_{ij} + 4\alpha^2 a_i\ a_j\Big], \\
Q_iQ_j - Q_{34,i}Q_{34,j} &\rightarrow \frac{1}{8\gamma}\ \delta_{ij}.
\end{aligned} \tag{B11}$$

Here, $\Rightarrow$ means the limit $\mathbf{a} = \mathbf{q}$. Also, above we have left out terms quadratic in the momenta $\mathbf{q}$ and $\mathbf{k}$.

2. Overlaps for (56) → (56) Transitions

(a) $L_M = 0$: $\mathcal{Y}^1_{m_P}(\mathbf{k}_3 - \mathbf{k}_5) = \sqrt{\frac{3}{4\pi}}\ i\boldsymbol{\epsilon}\cdot(2\mathbf{Q}_{34} - \mathbf{Q} - \mathbf{k}) \Rightarrow$

$$I_m^{(L=0)}(\mathbf{k}^2) = i\sqrt{\frac{3}{4\pi}}\ \boldsymbol{\epsilon}_{m,i}\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2} \int d^3Q_{34} \int d^3Q\ \left(2Q_{34,i} - Q_i - k_i\right)\cdot$$

$$\times \exp\Big\{-(9\alpha+4\gamma)\mathbf{Q}_{34}^2 + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34} - 4\gamma\mathbf{Q}^2 + 8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34}\Big\}$$

$$
\begin{aligned}
&= i\sqrt{\frac{3}{4\pi}}\ \boldsymbol{\epsilon}_{m,i}\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{4\gamma}\right)^{3/2}\int d^3Q_{34}\ \left(Q_{34,i}-k_i\right)\cdot \\
&\quad\times\exp\Big\{-9\alpha\mathbf{Q}_{34}^2+6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}\Big\} \\
&= i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{4\gamma}\right)^{3/2}\ \left(-\frac{1}{6}-1\right)\left(\frac{\pi}{9\alpha}\right)^{3/2}\exp(\alpha\mathbf{q}^2). \\
&= i\sqrt{\frac{3}{4\pi}}(\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\left(-\frac{7}{6}\right)\underbrace{\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}}\exp(\alpha\mathbf{q}^2). \qquad \text{(B12)}
\end{aligned}
$$

Here, we used that $\mathbf{q}=-\mathbf{k}/2$. With

$$
\mathcal{N}_0=\left(\frac{\sqrt{3}R_A^2}{\pi}\right)^3\left(\frac{R_M^2}{\pi}\right)^{3/4}=\left(\frac{6\sqrt{3}\alpha}{\pi}\right)^3\left(\frac{8\gamma}{\pi}\right)^{3/4} \qquad \text{(B13)}
$$

we have

$$
\begin{aligned}
I_m^{(L=0)}(\mathbf{k}^2=0) &= i\sqrt{\frac{3}{4\pi}}(\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\left(-\frac{7}{6}\right)\ \underbrace{2\sqrt{2}\left(\frac{\pi}{8\gamma}\right)^{3/4}} \\
&= -\frac{7}{3}\sqrt{2}\ \pi^{3/4}\ R_M^{-3/2}\ i\sqrt{\frac{3}{4\pi}}(\boldsymbol{\epsilon}_m\cdot\mathbf{k}). \qquad \text{(B14)}
\end{aligned}
$$

For future reference we define

$$
\bar{I}_0(\mathbf{k}^2=0)\ =\ \underbrace{2\sqrt{2}\left(\frac{\pi}{8\gamma}\right)^{3/4}}=2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4}. \qquad \text{(B15)}
$$

b. $\underline{L_M=1}$: This case in the momentum-space overlap-integral the integrand contains the factor

$$
\begin{aligned}
&\mathcal{Y}^1_{m_P}(\mathbf{k}_4-\mathbf{k}_5)(\mathcal{Y}_M)^{1*}_{m_M}(\mathbf{k}_3'-\mathbf{k}_5)\Rightarrow \\
&-\frac{3}{4\pi}(\boldsymbol{\epsilon}_{m_P}\cdot(2\mathbf{Q}_{34}-\mathbf{Q}-\mathbf{k}))\ (\boldsymbol{\epsilon}_{m_M}\cdot(2\mathbf{Q}_{34}-2\mathbf{Q}))\ =-\frac{3}{4\pi}(\boldsymbol{\epsilon}_{m_P})_i(\boldsymbol{\epsilon}_{m_M})_j\cdot \\
&\times\Big[4Q_{34,i}Q_{34,j}-4Q_{34,i}Q_j-2Q_iQ_{34.j}+2Q_iQ_j-2k_i\ Q_{34,j}+2k_iQ_j\Big] \\
&\Rightarrow\frac{1}{4\gamma}\delta_{ij}+O(q^2,qk),
\end{aligned}
$$

which yields, neglecting q^2-, qk-terms,

$$
\begin{aligned}
I_{m,n}^{(L=1)}(\mathbf{k}^2=0) &= -\frac{3}{4\pi\sqrt{2}}R_M\ (\boldsymbol{\epsilon}_m)_i(\boldsymbol{\epsilon}_n^*)_j\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\int d^3Q_{34}\int d^3Q \\
&\quad\times(2Q_iQ_j-2Q_{34,i}Q_{34,j})\cdot \\
&\quad\times\exp\Big\{-(9\alpha+4\gamma)\mathbf{Q}_{34}^2+6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}-4\gamma\mathbf{Q}^2+8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34}\Big\}
\end{aligned}
$$

$$\Rightarrow -\frac{3}{4\pi\sqrt{2}} R_M \, (\boldsymbol{\epsilon}_m \cdot \boldsymbol{\epsilon}_n^*) \cdot \underbrace{\mathcal{N}_0 \left(\frac{1}{8}\right) \left(\frac{\pi}{3\alpha}\right)^{3/2} \left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}} \cdot \frac{1}{4\gamma}$$

$$\Rightarrow -\frac{3}{4\pi} \, (\boldsymbol{\epsilon}_m \cdot \boldsymbol{\epsilon}_n^*) \cdot 4 \left(\frac{\pi}{R_M^2}\right)^{3/4} R_M^{-1} . \tag{B16}$$

c. Summary: The $(\underline{56}) \rightarrow (\underline{56})$ transition momentum-space overlap-integral, in the special frame where $\mathbf{p} = 0$, $\mathbf{q} = -\mathbf{k}/2$, neglecting $\mathbf{k}^2, qk$-terms, we obtain

$$a. \; I_m^{(L=0)}(\mathbf{k}^2 = 0) \; = \; -\frac{7}{3}\sqrt{2} \left(\frac{\pi}{R_M^2}\right)^{3/4} \cdot i\sqrt{\frac{3}{4\pi}} (\boldsymbol{\epsilon}_m \cdot \mathbf{k}), \tag{B17}$$

$$b. \; I_{m,n}^{(L=1)}(\mathbf{k}^2 = 0) \; = \; -4 \left(\frac{\pi}{R_M^2}\right)^{3/4} R_M^{-1} \cdot \frac{3}{4\pi} (\boldsymbol{\epsilon}_m \cdot \boldsymbol{\epsilon}_n^*), \tag{B18}$$

where

$$(\boldsymbol{\epsilon}_m \cdot \boldsymbol{\epsilon}_n^*) = \delta_{m,n} \, .$$

APPENDIX C: SU(6)-WAVE FUNCTIONS AND JACOBIAN-COORDINATES

The Jacobian-coordinates for a 3-body sytem are

$$\mathbf{k}_\rho = \frac{1}{\sqrt{2}} (\mathbf{k}_1 - \mathbf{k}_2) \quad , \quad \mathbf{k}_\lambda = \frac{1}{\sqrt{6}} (\mathbf{k}_1 + \mathbf{k}_2 - 2\mathbf{k}_3). \tag{C1}$$

The momentum-space the wave functions are

$$\begin{aligned} |(\underline{56}, L = 0^+), \{8\}, P_{11}\rangle \; &: \; \widetilde{\psi}_{N=0} = N_0 \; \exp\left[-\frac{1}{2} R_N^2 \left(\mathbf{k}_\rho^2 + \mathbf{k}_\lambda^2\right)\right], \\ |(\underline{70}, L = 0^+), \{8\}, P_{11}\rangle'' \; &: \; \widetilde{\psi}''_{N=1} = N_1 \; \left(\mathbf{k}_\rho^2 - \mathbf{k}_\lambda^2\right) \; \exp\left[-\frac{1}{2} R_N^2 \left(\mathbf{k}_\rho^2 + \mathbf{k}_\lambda^2\right)\right], \\ |(\underline{70}, L = 0^+), \{8\}, P_{11}\rangle' \; &: \; \widetilde{\psi}'_{N=1} = N_1 \; (2\mathbf{k}_\rho \cdot \mathbf{k}_\lambda) \; \exp\left[-\frac{1}{2} R_N^2 \left(\mathbf{k}_\rho^2 + \mathbf{k}_\lambda^2\right)\right]. \end{aligned} \tag{C2}$$

Here, the double-primed and the single-primed functions are antisymmetric respectively symmetric w.r.t. two indices, which we choose, without loss of generality, to be 1 and 2. The normalization constants are

$$N_0 = \left(\frac{\sqrt{3} R_A^2}{\pi}\right)^{3/2} \quad , \quad N_1 = (R_A^2/\sqrt{3}) \left(\frac{\sqrt{3} R_A^2}{\pi}\right)^{3/2} . \tag{C3}$$

In terms of the momenta of the previous sections, we have

$$\begin{aligned} \mathbf{k}_\rho \; &= \; \frac{1}{\sqrt{2}} \boldsymbol{\Delta}_{12} \; , \; \mathbf{k}_\lambda = \frac{1}{\sqrt{6}} (\mathbf{q} - 3\mathbf{Q}_{34} - \mathbf{k}) \equiv \frac{1}{\sqrt{6}} (\mathbf{u} - 3\mathbf{Q}_{34}), \\ \mathbf{k}'_\rho \; &= \; \frac{1}{\sqrt{2}} \boldsymbol{\Delta}_{12} \; , \; \mathbf{k}'_\lambda = \frac{1}{\sqrt{6}} (\mathbf{q} - 3\mathbf{Q}_{34} + \mathbf{k}) \equiv \frac{1}{\sqrt{6}} (\mathbf{w} - 3\mathbf{Q}_{34}). \end{aligned} \tag{C4}$$

Now, for $\mathbf{k} = 0$

$$\begin{aligned}
\mathbf{k}_\rho \cdot \mathbf{k}_\lambda &= \frac{1}{2\sqrt{3}} \left(\mathbf{u} - 3\mathbf{Q}_{34}\right) \cdot \boldsymbol{\Delta}_{12}, \\
\mathbf{k}_\rho^2 - \mathbf{k}_\lambda^2 &= \frac{1}{2}\boldsymbol{\Delta}_{12}^2 - \frac{1}{6}\left(\mathbf{u}^2 - 6\mathbf{u}\cdot\mathbf{Q}_{34} + 9\mathbf{Q}_{34}^2\right). \qquad \text{(C5)}
\end{aligned}$$

we recall that

$$\begin{aligned}
f(\mathbf{k}_1, ..., \mathbf{k}_4) &= -\alpha(\mathbf{q}^2 + \mathbf{k}^2) + \bar{f}(\mathbf{k}_1, ..., \mathbf{k}_4), \\
\bar{f}(\mathbf{k}_1, ..., \mathbf{k}_4) &= -3\alpha\ \boldsymbol{\Delta}_{12}^2 - (9\alpha + 4\gamma)\mathbf{Q}_{34}^2 + 6\alpha\mathbf{q}\cdot\mathbf{Q}_{34} + 8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34} - 4\gamma\mathbf{Q}^2.
\end{aligned}$$

Then, we get the integrals

$$\begin{aligned}
\int d^3k_1 d^3k_2 &\Rightarrow \frac{1}{8}\int d^3\Delta_{12} \Rightarrow \frac{1}{8}\left(\frac{\pi}{3\alpha}\right)^{3/2}, \\
\int d^3k_1 d^3k_2 \boldsymbol{\Delta}_{12}^2 &\Rightarrow -\frac{1}{3}\frac{d}{d\alpha} \Rightarrow \frac{1}{2\alpha}\cdot\frac{1}{8}\left(\frac{\pi}{3\alpha}\right)^{3/2}.
\end{aligned}$$

For the overlap between the SU(6)-irreps (56)- and (70)-irreps we have for the (70) in the initial state

$$\mathbf{k}_\rho^2 - \mathbf{k}_\lambda^2 = \frac{1}{2}\boldsymbol{\Delta}_{12}^2 - \frac{1}{6}\left(\mathbf{u}^2 - 6\mathbf{u}\cdot\mathbf{Q}_{34} + 9\mathbf{Q}_{34}^2\right), \qquad \text{(C6)}$$

where $\mathbf{u} \equiv \mathbf{q} - \mathbf{k}$. For the (70) in the final state we have the factor $\mathbf{k}_\rho^2 - \mathbf{k'}_\lambda^2$ which is the same as in (C6) apart from the replacement $\mathbf{u} \to \mathbf{w} = \mathbf{q} + \mathbf{k}$.

The basic overlap integral with the $\bar{f}(\mathbf{k}_1, ..., \mathbf{k}_4)$-function, as seen from (B10), is

$$\begin{aligned}
\bar{I}_0(A; B, M) &= \mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2} \exp\left[\alpha\mathbf{a}^2\right] \\
&\Rightarrow 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} = 2\sqrt{2}\ \pi^{3/4}\ R_M^{-3/2}, \qquad \text{(C7)}
\end{aligned}$$

For the computation of the momentum-space overlaps we need the results:

$$\begin{aligned}
(\Delta_{12})_i &\longrightarrow 0 \quad , \quad (Q_{34})_i \to +\frac{1}{6\alpha}\nabla_{q,i} \to \frac{2\alpha}{6\alpha}\ q_i, \\
(Q_{34})_i(Q_{34})_j &\to +\frac{1}{36\alpha^2}\nabla_{q,i}\nabla_{q,j} \to \frac{1}{36\alpha^2}\left[2\alpha\delta_{ij} + 4\alpha^2 q_i q_j\right].
\end{aligned}$$

In passing we note that from above it follows that for the SU(6)-irrep transition overlaps

$$I(\psi^s|\psi', M) = 0\ ,\ I(\psi^s|\psi'', M) \neq 0. \qquad \text{(C8)}$$

With these results, we readily derive that

$$\mathbf{k}_\rho^2 - \mathbf{k}_\lambda^2 \Rightarrow \frac{1}{4\alpha} - \frac{9}{6}(6\alpha)^{-1} \Rightarrow 0, \qquad \text{(C9)}$$

which is logical because the $\underline{(56)}$ and $\underline{(70)}$ are orthogonal.
For the normalization constant of the $\underline{(70)}$-irrep

$$\mathcal{N}_0'' = N_0\ N_1\ \left(\frac{R_M^2}{\pi}\right)^{3/4} = (R_A^2/\sqrt{3})\ \mathcal{N}_0 \tag{C10}$$

For the finally needed overlap-integrals we have extra factors Q_{34} in the integrands. Therefore, we need

$$\begin{aligned}(Q_{34})_i(Q_{34})_j(Q_{34})_k\ &\Rightarrow +\left(\frac{1}{6\alpha}\right)^3 (4\alpha^2)\left(\delta_{ij}\ q_k + \delta_{ik}\ q_j + \delta_{jk}\ q_i\right)\\ (Q_{34})_i(Q_{34})_j(Q_{34})_k(Q_{34})_l\ &\rightarrow\ \left(\frac{1}{6\alpha}\right)^4 (4\alpha^2)\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right),\end{aligned} \tag{C11}$$

where in the last expression we restricted ourselves to the limit $\mathbf{k}^2 = 0$.

1. Overlaps for $\underline{(56)} \rightarrow \underline{(70)}$ Transitions

(a) $\underline{L_M = 0}$: $\mathcal{Y}^1_{m_P}(\mathbf{k}_3 - \mathbf{k}_5) = \sqrt{\frac{3}{4\pi}}\ i\boldsymbol{\epsilon}\cdot(2\mathbf{Q}_{34} - \mathbf{Q} - \mathbf{k}) \Rightarrow$

$$\begin{aligned}I_m^{(L=0)\prime\prime}(\mathbf{k}^2)\ &=\ i\sqrt{\frac{3}{4\pi}}\boldsymbol{\epsilon}_{m,i}\cdot\mathcal{N}_0''\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\int d^3Q_{34}\ \int d^3Q\ \left(2Q_{34,i} - Q_i - k_i\right)\cdot\\ &\times\left\{\frac{1}{4\alpha} - \frac{1}{6}\left(\mathbf{w}^2 - 6\mathbf{w}\cdot\mathbf{Q}_{34} + 9\mathbf{Q}_{34}^2\right)\right\}\cdot\\ &\times\exp\left\{-(9\alpha+4\gamma)\mathbf{Q}_{34}^2 + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34} + 8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34} - 4\gamma\mathbf{Q}_{34}^2\right\}\\ &= i\sqrt{\frac{3}{4\pi}}\boldsymbol{\epsilon}_{m,i}\cdot\mathcal{N}_0''\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{4\gamma}\right)^{3/2}\int d^3Q_{34}\ Q_{34,i}\cdot\\ &\times\left\{\frac{1}{4\alpha} - \frac{9}{6}\mathbf{Q}_{34}^2\right\}\ \exp\left\{-9\alpha\mathbf{Q}_{34}^2 + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}\right\}\\ &= -i\sqrt{\frac{3}{4\pi}}\left(\boldsymbol{\epsilon}_m\cdot\mathbf{k}\right)\ \mathcal{N}_0''\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}\ \frac{1}{36\alpha},\end{aligned} \tag{C12}$$

where we used $\mathbf{a} = \mathbf{q} = -\mathbf{k}/2$. From this we obtain

$$\begin{aligned}I_m^{(L=0)\prime\prime}(\mathbf{k}^2)\ &=\ -i\sqrt{\frac{3}{4\pi}}\ \left(\boldsymbol{\epsilon}_m\cdot\mathbf{k}\right)\cdot\frac{R_A^2}{\sqrt{3}}\ \underbrace{\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}}\ \frac{1}{36\alpha}\\ &=\ -i\sqrt{\frac{3}{4\pi}}\ \left(\boldsymbol{\epsilon}_m\cdot\mathbf{k}\right)\cdot\frac{1}{3}\sqrt{\frac{2}{3}}\left(\frac{\pi}{R_M^2}\right)^{3/4}.\end{aligned} \tag{C13}$$

b. $\underline{L_M = 1}$: This case in the momentum-space overlap-integral the integrand contains the factor

$$\begin{aligned}&\mathcal{Y}^1_{m_P}(\mathbf{k}_4 - \mathbf{k}_5)(\mathcal{Y}_M)^{1*}_{m_M}(\mathbf{k}_3' - \mathbf{k}_5) \Rightarrow\\ &-\frac{3}{4\pi}\boldsymbol{\epsilon}_{m_P}\cdot(2\mathbf{Q}_{34} - \mathbf{Q} - \mathbf{k})\ \boldsymbol{\epsilon}^*_{m_M}\cdot(2\mathbf{Q}_{34} - 2\mathbf{Q}) = -\frac{3}{4\pi}(\boldsymbol{\epsilon}_{m_P})_i(\boldsymbol{\epsilon}_{m_M})_j\cdot\\ &\times\left[4Q_{34,i}Q_{34,j} - 4Q_{34}Q_j - 2Q_iQ_{34,j} + 2Q_iQ_j - 2k_i\ Q_{34,j} + 2k_iQ_j\right],\end{aligned}$$

which gives, using the computed integrals,

$$
\begin{aligned}
I^{(L=1)\prime\prime}_{m,n}(\mathbf{k}^2=0) \;\Rightarrow\; & -\frac{3}{4\pi\sqrt{2}} R_M\ (\boldsymbol{\epsilon}_m)_i(\boldsymbol{\epsilon}^*_n)_j \cdot \mathcal{N}_0^{\prime\prime} \left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2} \int d^3Q_{34} \int d^3Q \\
& \times (2Q_iQ_j - 2Q_{34,i}Q_{34,j}) \cdot \left\{\frac{1}{4\alpha} - \frac{1}{6}\left(\mathbf{w}^2 - 6\mathbf{w}\cdot\mathbf{Q}_{34} + 9\mathbf{Q}_{34}^2\right)\right\} \cdot \\
& \times \exp\left\{-(9\alpha+4\gamma)\mathbf{Q}_{34}^2 + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34} - 4\gamma\mathbf{Q}^2 + 8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34}\right\} \\
= \; & -\frac{3}{4\pi\sqrt{2}} R_M\ (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}^*_n) \cdot \mathcal{N}_0^{\prime\prime} \left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2} \left(\frac{\pi}{4\gamma}\right)^{3/2} \cdot \frac{1}{4\gamma} \cdot \\
& \times \int d^3Q_{34} \left(\frac{1}{4\alpha} - \frac{9}{6}\mathbf{Q}_{34}^2\right) \exp\left\{-9\alpha\mathbf{Q}_{34}^2 + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}\right\} = 0(!?)
\end{aligned}
\tag{C14}
$$

c. Summary: The ($\underline{56}$) → ($\underline{70}$) transition momentum-space overlap-integral, in the special frame where $\mathbf{p}=0$, $\mathbf{q}=-\mathbf{k}/2$, neglecting $\mathbf{k}^2$-terms, we obtain

$$
\begin{aligned}
a.\ I^{\prime\prime(L=0)}_m(\mathbf{k}) \;&=\; -i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k}) \cdot \frac{1}{3}\sqrt{\frac{2}{3}}\left(\frac{\pi}{R_M^2}\right)^{3/4}, \\
b.\ I^{(L=1)\prime\prime}_{m,n}(\mathbf{k}) \;&=\; -\frac{3}{4\pi}\ (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}^*_n)\frac{2R_A^2}{\sqrt{3}R_M}\left(\frac{\pi}{R_M^2}\right)^{3/4} \cdot \left\{\frac{3}{R_A^2} - \frac{3}{R_A^2}\right\} = 0.
\end{aligned}
$$

where is used

$$(\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}^*_n) = \delta_{m,n}\ .$$

APPENDIX D: QUARK FORM-FACTORS

In this appendix we introduce a quark form factor in the overlap integrals. It appeared above that for the L=1 mesons there is no transition matrix element between the SU6-irreps ($\underline{56}$) vand ($\underline{70}$). This occurred because of a subtile cancellation. To disrupt this cancellation we introduce a cut-off. For this here a convenient choice is a gaussian form factor $F(\mathbf{Q}^2) = \exp[-4\lambda\mathbf{Q}^2]$ in order to generate a non-zero transiton matrix element. Similarly as before, we now have

$$
\begin{aligned}
\bar{I}_0 \;&=\; \mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2} \int d^3Q_{34} \int d^3Q\ \exp\left\{-(9\alpha+4\gamma)\mathbf{Q}_{34}^2\right. \\
&\qquad \left. - 4(\gamma+\lambda)\mathbf{Q}^2 + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34} + 8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34}\right\} \\
&=\; \mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{4(\gamma+\lambda)}\right)^{3/2} \int d^3Q_{34} \cdot \\
&\qquad \times \exp\left\{-\left((9\alpha+4\gamma) - \frac{4\gamma^2}{\gamma+\lambda}\right)\mathbf{Q}_{34}^2 + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{N}_0 \left(\frac{1}{8}\right) \left(\frac{\pi}{3\alpha}\right)^{3/2} \left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2} \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2} \exp\left[\bar{\alpha}\mathbf{a}^2\right] \\
&\Rightarrow 2\sqrt{2}\left(\frac{\pi}{R_M^2}\right)^{3/4} \left(\frac{3R_A^2R_M^2}{3R_A^2(R_M^2+R_\lambda^2)+R_M^2R_\lambda^2}\right)^{3/2}, \qquad \text{(D1)}
\end{aligned}
$$

where $\mathbf{a} \equiv \mathbf{q}$ and

$$
\bar{\alpha} = \frac{9\alpha(\gamma+\lambda)}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\ \alpha = \left[1+\frac{4\gamma}{9\alpha}\frac{\lambda}{(\gamma+\lambda)}\right]^{-1} \alpha. \qquad \text{(D2)}
$$

Above, we performed with $\mathbf{b} \equiv \mathbf{Q}_{34}$ the integral

$$
\int d^3Q \exp\Big[-4(\gamma+\lambda)\mathbf{Q}^2+8\gamma\mathbf{b}\cdot\mathbf{Q}\Big] = \left(\frac{\pi}{4(\gamma+\lambda)}\right)^{3/2} \exp\left(\frac{4\gamma^2}{\gamma+\lambda}\mathbf{b}^2\right),
$$

from which we derive that extra Q-vector components in the integrand lead

$$
\begin{aligned}
Q_i \to (1/8\gamma)\nabla_{b,i} &\to \frac{\gamma}{\gamma+\lambda} b_i \left(\frac{\pi}{4(\gamma+\lambda)}\right)^{3/2} \exp\left(\frac{4\gamma^2}{\gamma+\lambda}\mathbf{b}^2\right), \\
Q_iQ_j \to (1/8\gamma)^2\nabla_{b,i}\nabla_{b,j} &\to \left[\frac{1}{8(\gamma+\lambda)}\delta_{ij} + \left(\frac{\gamma}{\gamma+\lambda}\right)^2 b_ib_j\right] \cdot \\
&\quad \times \left(\frac{\pi}{4(\gamma+\lambda)}\right)^{3/2} \exp\left(\frac{4\gamma^2}{\gamma+\lambda}\mathbf{b}^2\right).
\end{aligned}
$$

With extra components of $\mathbf{Q}$ in the integrand, the integrals get the following factors w.r.t. the basic integral $\bar{I}_0$:

$$
\begin{aligned}
Q_{34,i} &\to \frac{1}{6\alpha}\boldsymbol{\nabla}_{a,i} \to \frac{2\bar{\alpha}}{6\alpha}\ a_i, \\
Q_{34,i}Q_{34,j} &\to \left(\frac{1}{6\alpha}\right)^2 \boldsymbol{\nabla}_{a,i}\boldsymbol{\nabla}_{a,j} \to \left(\frac{1}{6\alpha}\right)^2 \left[2\bar{\alpha}\ \delta_{ij} + 4\bar{\alpha}^2 a_i\ a_j\right], \\
Q_{34_i}Q_{34,j}Q_{34,k} &\to \left(\frac{1}{6\alpha}\right)^3 \boldsymbol{\nabla}_{a,i}\boldsymbol{\nabla}_{a,j}\boldsymbol{\nabla}_{a,k} \to \left(\frac{1}{6\alpha}\right)^3 \left[4\bar{\alpha}^2\left(\delta_{ij}\ a_k + \delta_{ik}\ a_j + \delta_{jk}\ a_i\right)\right], \\
Q_{34_i}Q_{34,j}Q_{34,k}Q_{34,l} &\to \left(\frac{1}{6\alpha}\right)^4 \boldsymbol{\nabla}_{a,i}\boldsymbol{\nabla}_{a,j}\boldsymbol{\nabla}_{a,k}\boldsymbol{\nabla}_{a,l} \to \left(\frac{1}{6\alpha}\right)^4 \left[4\bar{\alpha}^2\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right)\right], \\
Q_{34,i}Q_j &\to \frac{\gamma}{\gamma+\lambda}\ \left(\frac{1}{6\alpha}\right)^2 \left[2\bar{\alpha}\ \delta_{ij} + 4\bar{\alpha}^2 a_i\ a_j\right], \\
Q_iQ_j - Q_{34,i}Q_{34,j} &\to \frac{1}{8(\gamma+\lambda)}\ \delta_{ij} - \left[1-\left(\frac{\gamma}{\gamma+\lambda}\right)^2\right]\ Q_{34,i}Q_{34,j} \\
&= \frac{1}{8(\gamma+\lambda)}\left\{\delta_{ij} - 8\lambda\frac{2\gamma+\lambda}{\gamma+\lambda}\ Q_{34,i}Q_{34,j}\right\}. \qquad \text{(D3)}
\end{aligned}
$$

Here, $\Rightarrow$ means the limit $\mathbf{a} = \mathbf{q}$. Also, above we have left out terms quadratic in the momenta $\mathbf{q}$ and $\mathbf{k}$.

1. Overlaps for (<u>56</u>) → (<u>56</u>) Transitions

(a) $\underline{L_M = 0}$: $\mathcal{Y}^1_{m_P}(\mathbf{k}_3 - \mathbf{k}_5) = \sqrt{\frac{3}{4\pi}}\ i\boldsymbol{\epsilon}\cdot(2\mathbf{Q}_{34} - \mathbf{Q} - \mathbf{k}) \Rightarrow$

$$I^{(L=0)}_m(\mathbf{k}^2) = i\sqrt{\frac{3}{4\pi}}\ \boldsymbol{\epsilon}_{m,i}\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\ \int d^3Q_{34}\ \int d^3Q\ \ (2Q_{34,i} - Q_i - k_i)\cdot$$
$$\times\exp\Big\{ - (9\alpha+4\gamma)\mathbf{Q}^2_{34} + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34} - 4(\gamma+\lambda)\mathbf{Q}^2 + 8\gamma\mathbf{Q}\cdot\mathbf{Q}_{34}\Big\}$$
$$= i\sqrt{\frac{3}{4\pi}}\ \boldsymbol{\epsilon}_{m,i}\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{4(\gamma+\lambda)}\right)^{3/2}\int d^3Q_{34}\ \left(\frac{\gamma+2\lambda}{\gamma+\lambda}\ Q_{34,i} - k_i\right)\cdot$$
$$\times\exp\Big\{ - \left((9\alpha+4\gamma) - \frac{4\gamma^2}{\gamma+\lambda}\right)\mathbf{Q}^2_{34} + 6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}\Big\}$$
$$= i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}\ \left[-\frac{\gamma+2\lambda}{\gamma+\lambda}\ \frac{\bar{\alpha}}{6\alpha} - 1\right]\cdot$$
$$\times\left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}\exp(\alpha\mathbf{q}^2)$$
$$= i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\left[-\frac{\gamma+2\lambda}{\gamma+\lambda}\ \frac{\bar{\alpha}}{6\alpha} - 1\right]2\sqrt{2}\left(\frac{\pi}{8\gamma}\right)^{3/4}\ \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}\ \exp(\alpha\mathbf{q}^2). \tag{D4}$$

Here, we used that $\mathbf{q} = -\mathbf{k}/2$. We have

$$I^{(L=0)}_m(\mathbf{k}^2 = 0)\ =\ -i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\left[\frac{\gamma+2\lambda}{\gamma+\lambda}\ \frac{\bar{\alpha}}{6\alpha} + 1\right]2\sqrt{2}\left(\frac{\pi}{8\gamma}\right)^{3/4}\cdot$$
$$\times\left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}. \tag{D5}$$

(b) $\underline{L_M = 1}$:

$$\mathcal{Y}^1_{m_P}(\mathbf{k}_4 - \mathbf{k}_5)(\mathcal{Y}_M)^{1*}_{m_M}(\mathbf{k}'_3 - \mathbf{k}_5) \Rightarrow$$
$$-\frac{3}{4\pi}(\boldsymbol{\epsilon}_{m_P}\cdot(2\mathbf{Q}_{34} - \mathbf{Q} - \mathbf{k}))\ (\boldsymbol{\epsilon}_{m_M}\cdot(2\mathbf{Q}_{34} - 2\mathbf{Q}))\ \ = -\frac{3}{4\pi}(\boldsymbol{\epsilon}_{m_P})_i(\boldsymbol{\epsilon}_{m_M})_j\cdot$$
$$\times\Big[4Q_{34,i}Q_{34,j} - 4Q_{34,i}Q_j - 2Q_iQ_{34.j} + 2Q_iQ_j - 2k_i\ Q_{34,j} + 2k_iQ_j\Big]$$
$$\Rightarrow\frac{1}{8(\gamma+\lambda)}\ \delta_{ij} - \left[1 - \left(\frac{\gamma}{\gamma+\lambda}\right)^2\right]\ Q_{34,i}Q_{34,j}.$$

$$I^{(L=1)}_{m,n}(\mathbf{k}^2) = -\frac{3}{4\pi\sqrt{2}}\ R_M\ \boldsymbol{\epsilon}_{m,i}\boldsymbol{\epsilon}_{n,j}\cdot\mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\ \int d^3Q_{34}\ \int d^3Q\cdot$$

$$
\times \left\{ \frac{1}{8(\gamma+\lambda)} \; \delta_{ij} - \left[1 - \left(\frac{\gamma}{\gamma+\lambda} \right)^2 \right] \; Q_{34,i} Q_{34,j} \right\} \cdot
$$

$$
\times \exp \left\{ - (9\alpha + 4\gamma) \mathbf{Q}_{34}^2 + 6\alpha \mathbf{a} \cdot \mathbf{Q}_{34} - 4(\gamma+\lambda) \mathbf{Q}^2 + 8\gamma \mathbf{Q} \cdot \mathbf{Q}_{34} \right\}
$$

$$
= - \frac{3}{4\pi\sqrt{2}} \; R_M \; (\boldsymbol{\epsilon}_m \cdot \boldsymbol{\epsilon}_n) \cdot \mathcal{N}_0 \left(\frac{1}{8} \right) \left(\frac{\pi}{3\alpha} \right)^{3/2} \; \left(\frac{\pi^2}{36\alpha\gamma} \right)^{3/2} \cdot
$$

$$
\times \left\{ \frac{1}{8(\gamma+\lambda)} - \left[1 - \left(\frac{\gamma}{\gamma+\lambda} \right)^2 \right] \; \left(\frac{2\bar{\alpha}}{36\alpha^2} \right) \right\} \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda) + 4\gamma\lambda} \right)^{3/2} \exp(\alpha \mathbf{q}^2)
$$

$$
= - \frac{3}{4\pi\sqrt{2}} \; R_M \; (\boldsymbol{\epsilon}_m \cdot \boldsymbol{\epsilon}_n) \cdot 2\sqrt{2} \left(\frac{\pi}{8\gamma} \right)^{3/4} \left\{ \frac{1}{8(\gamma+\lambda)} - \left[1 - \left(\frac{\gamma}{\gamma+\lambda} \right)^2 \right] \; \left(\frac{2\bar{\alpha}}{36\alpha^2} \right) \right\}
$$

$$
\times \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda) + 4\gamma\lambda} \right)^{3/2} \exp(\alpha \mathbf{q}^2). \tag{D6}
$$

we have

$$
\begin{aligned}
I^{(L=1)}_{m,n}(\mathbf{k}^2=0) \; = \; & - \frac{3}{4\pi\sqrt{2}} \; R_M \; (\boldsymbol{\epsilon}_m \cdot \boldsymbol{\epsilon}_n) \cdot 2\sqrt{2} \left(\frac{\pi}{8\gamma} \right)^{3/4} \; \frac{1}{8\gamma} \cdot \\
& \times \left\{ \frac{\gamma}{(\gamma+\lambda)} - \left[1 - \left(\frac{\gamma}{\gamma+\lambda} \right)^2 \right] \; \left(\frac{4\bar{\alpha}\gamma}{9\alpha^2} \right) \right\} \cdot \\
& \times \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda) + 4\gamma\lambda} \right)^{3/2} .
\end{aligned} \tag{D7}
$$

2. Overlaps for $(\underline{56}) \to (\underline{70})$ Transitions

(a) $\underline{L_M = 0}$: $\mathcal{Y}^1_{m_P}(\mathbf{k}_3 - \mathbf{k}_5) = \sqrt{\frac{3}{4\pi}} \; i \boldsymbol{\epsilon} \cdot (2\mathbf{Q}_{34} - \mathbf{Q} - \mathbf{k}) \Rightarrow$

$$
I^{(L=0)\prime\prime}_{m}(\mathbf{k}^2) = i \sqrt{\frac{3}{4\pi}} \; \boldsymbol{\epsilon}_{m,i} \cdot \mathcal{N}_0^{\prime\prime} \left(\frac{1}{8} \right) \left(\frac{\pi}{3\alpha} \right)^{3/2} \; \int d^3Q_{34} \; \int d^3Q
$$

$$
\times (2Q_{34,i} - Q_i - k_i) \cdot \left\{ \frac{1}{4\alpha} - \frac{1}{6} \left(\mathbf{w}^2 - 6\mathbf{w} \cdot \mathbf{Q}_{34} + 9\mathbf{Q}_{34}^2 \right) \right\} \cdot
$$

$$
\times \exp \left\{ - (9\alpha + 4\gamma) \mathbf{Q}_{34}^2 + 6\alpha \mathbf{a} \cdot \mathbf{Q}_{34} - 4(\gamma+\lambda) \mathbf{Q}^2 + 8\gamma \mathbf{Q} \cdot \mathbf{Q}_{34} \right\}
$$

$$
= i \sqrt{\frac{3}{4\pi}} \; \boldsymbol{\epsilon}_{m,i} \cdot \mathcal{N}_0^{\prime\prime} \left(\frac{1}{8} \right) \left(\frac{\pi}{3\alpha} \right)^{3/2} \left(\frac{\pi}{4(\gamma+\lambda)} \right)^{3/2} \int d^3Q_{34} \; \left(\frac{\gamma + 2\lambda}{\gamma+\lambda} \; Q_{34,i} - k_i \right) \cdot
$$

$$
\times \left\{ \frac{1}{4\alpha} - \frac{1}{6} \left(\mathbf{w}^2 - 6\mathbf{w} \cdot \mathbf{Q}_{34} + 9\mathbf{Q}_{34}^2 \right) \right\} \cdot
$$

$$
\times \exp \left\{ - \left((9\alpha + 4\gamma) - \frac{4\gamma^2}{\gamma+\lambda} \right) \mathbf{Q}_{34}^2 + 6\alpha \mathbf{a} \cdot \mathbf{Q}_{34} \right\}
$$

$$
\begin{aligned}
&= i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\mathcal{N}_0''\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}\cdot \\
&\times\frac{1}{4\alpha}\left\{-\left[1+\frac{\bar{\alpha}}{6\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right]+\frac{\bar{\alpha}}{\alpha}\left(1+\frac{\bar{\alpha}}{18\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right)\right\}\cdot \\
&\times\left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}\exp(\alpha\mathbf{q}^2) \\
&= i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\frac{1}{4\alpha}\left\{-\left[1+\frac{\bar{\alpha}}{6\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right]+\frac{\bar{\alpha}}{\alpha}\left(1+\frac{\bar{\alpha}}{18\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right)\right\}\cdot \\
&\times 2\sqrt{3}\ \alpha\ 2\sqrt{2}\left(\frac{\pi}{8\gamma}\right)^{3/4}\left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}\exp(\alpha\mathbf{q}^2).
\end{aligned}
\tag{D8}
$$

Here, we used that $\mathbf{q}=-\mathbf{k}/2$. we have

$$
\begin{aligned}
&I_m^{(L=0)\prime\prime}(\mathbf{k}^2=0) = i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\left\{-\left[1+\frac{\bar{\alpha}}{6\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right]+\right. \\
&\left.+\frac{\bar{\alpha}}{\alpha}\left(1+\frac{\bar{\alpha}}{18\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right)\right\}\cdot \\
&\times\sqrt{6}\ \left(\frac{\pi}{8\gamma}\right)^{3/4}\left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}.
\end{aligned}
\tag{D9}
$$

(b) $\underline{L_M=1}$:

$$
\begin{aligned}
I_{m,n}^{(L=1)\prime\prime}(\mathbf{k}^2) &= -\frac{3}{4\pi\sqrt{2}}\ R_M\ \boldsymbol{\epsilon}_{m,i}\boldsymbol{\epsilon}_{n,j}\cdot\mathcal{N}_0''\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{4(\gamma+\lambda)}\right)^{3/2}\int d^3Q_{34}\cdot \\
&\times\left\{\frac{1}{8(\gamma+\lambda)}\delta_{ij}-\left[1-\left(\frac{\gamma}{\gamma+\lambda}\right)^2\right]\ Q_{34,i}Q_{34,j}\right\}\cdot \\
&\times\left\{\frac{1}{4\alpha}-\frac{1}{6}\left(\mathbf{w}^2-6\mathbf{w}\cdot\mathbf{Q}_{34}+9\mathbf{Q}_{34}^2\right)\right\}\cdot \\
&\times\exp\left\{-\left((9\alpha+4\gamma)-\frac{4\gamma^2}{\gamma+\lambda}\right)\mathbf{Q}_{34}^2+6\alpha\mathbf{a}\cdot\mathbf{Q}_{34}\right\} \\
&= -\frac{3}{4\pi\sqrt{2}}\ R_M\ (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n)\cdot\mathcal{N}_0''\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}\cdot \\
&\times\left\{\frac{1}{32\alpha(\gamma+\lambda)}\left(1-\frac{\bar{\alpha}}{\alpha}\right)-\frac{\bar{\alpha}}{72\alpha^3}\left(1-\frac{5}{3}\frac{\bar{\alpha}}{\alpha}\right)\left[1-\left(\frac{\gamma}{\gamma+\lambda}\right)^2\right]\right\}\cdot \\
&\times\left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}\exp(\alpha\mathbf{q}^2) \\
&= -\frac{3}{4\pi\sqrt{2}}\ R_M\ (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n)\cdot 2\sqrt{2}\left(\frac{\pi}{8\gamma}\right)^{3/4}\ \frac{R_A^2}{\sqrt{3}}\cdot
\end{aligned}
$$

$$\times \left\{ \frac{1}{32\alpha(\gamma+\lambda)} \left(1-\frac{\bar{\alpha}}{\alpha}\right) - \frac{\bar{\alpha}}{72\alpha^3}\left(1-\frac{5}{3}\frac{\bar{\alpha}}{\alpha}\right)\left[1-\left(\frac{\gamma}{\gamma+\lambda}\right)^2\right]\right\} \cdot$$

$$\times \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2} \exp(\alpha\mathbf{q}^2). \tag{D10}$$

This gives

$$I_{m,n}^{(L=1)\prime\prime}(\mathbf{k}^2=0) = -\frac{3}{4\pi\sqrt{2}} R_M \, (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n)\cdot 2\sqrt{2}\left(\frac{\pi}{8\gamma}\right)^{3/4} \frac{R_A^2}{\sqrt{3}} \cdot$$
$$\times \left\{ \frac{1}{32\alpha(\gamma+\lambda)} \left(1-\frac{\bar{\alpha}}{\alpha}\right) - \frac{\bar{\alpha}}{72\alpha^3}\left(1-\frac{5}{3}\frac{\bar{\alpha}}{\alpha}\right) \cdot\right.$$
$$\left.\times \left[1-\left(\frac{\gamma}{\gamma+\lambda}\right)^2\right]\right\} \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2} \tag{D11}$$

3. Summary and Ratio's for Overlaps

(a) Ratio's for $\underline{L_M = 0}$:

$$\frac{I_m^{(L=0)\prime\prime}(\mathbf{k}^2=0)}{I_m^{(L=0)}(\mathbf{k}^2=0)} = \frac{1}{2}\sqrt{3}\left\{1-\frac{\bar{\alpha}}{\alpha}\left(1+\frac{\bar{\alpha}}{18\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right)\left(1+\frac{\bar{\alpha}}{6\alpha}\frac{\gamma+2\lambda}{\gamma+\lambda}\right)^{-1}\right\}. \tag{D12}$$

(b) Ratio's for $\underline{L_M = 1}$:

$$\frac{I_{m,n}^{(L=1)\prime\prime}(\mathbf{k}^2=0)}{I_{m,n}^{(L=1)}(\mathbf{k}^2=0)} = \frac{1}{2}\sqrt{3}\left\{\left(1-\frac{\bar{\alpha}}{\alpha}\right) - \frac{4\bar{\alpha}\lambda}{9\alpha^2}\left(1-\frac{5\bar{\alpha}}{3\alpha}\right)\frac{2\gamma+\lambda}{\gamma+\lambda}\right\}\cdot$$
$$\times\left(1-\frac{4\bar{\alpha}\lambda}{9\alpha^2}\frac{2\gamma+\lambda}{\gamma+\lambda}\right)^{-1}. \tag{D13}$$

APPENDIX E: CONNECTION OVERLAP INTEGRALS LE YOUANC ET AL

Using the representation $\delta_\epsilon(x) = \exp\left[-x^2/\epsilon\right]/\sqrt{\pi\epsilon}$ we have

$$\delta^3(\mathbf{Q}) = \lim_{\epsilon\to 0}\,(\pi\epsilon)^{-3/2}\,\exp\left(-\mathbf{Q}^2/\epsilon\right) = \lim_{\lambda\to\infty}\left(\frac{4\lambda}{\pi}\right)^{3/2}\exp\left(-4\lambda\,\mathbf{Q}^2\right). \tag{E1}$$

To establish the connection, we write the quark-gluon form-factor of section D as

$$F(\mathbf{Q}^2) = \left(\frac{\pi}{4\lambda_0}\right)^{3/2}\cdot\left(\frac{4\lambda}{\pi}\right)^{3/2}\exp\left(-4\lambda\,\mathbf{Q}^2\right), \tag{E2}$$

and take $\lim_{\lambda\to\infty}$. First we note that, see definition (D2),

$$\lim_{\lambda\to\infty}\ \frac{\bar{\alpha}}{\alpha}=\frac{9\alpha}{9\alpha+4\gamma}. \tag{E3}$$

Furthermore,

$$\begin{aligned}\bar{I}_0 \ &\Rightarrow\ \mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi^2}{36\alpha\gamma}\right)^{3/2}\ \left(\frac{9\alpha\gamma}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right)^{3/2}\cdot\left(\frac{4\lambda}{\pi}\right)^{3/2}\\ &\stackrel{\lambda\to\infty}{\longrightarrow}\ \mathcal{N}_0\left(\frac{1}{8}\right)\left(\frac{\pi}{3\alpha}\right)^{3/2}\left(\frac{\pi}{9\alpha+4\gamma}\right)^{3/2}=\left(\frac{3R_A^2R_M}{3R_A^2+R_M^2}\right)^{3/2}.\end{aligned} \tag{E4}$$

Below, we apply this limit to the overlap integrals of the foregoing section.

1. Overlaps for (56) → (56) Transitions

From (D5) we get, after multiplication with $(4\lambda/\pi)^{3/2}$,

$$\begin{aligned}I_m^{(L=0)}(\mathbf{k}^2=0)\ &\stackrel{\lambda\to\infty}{\longrightarrow}\ -i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\ \left[2\sqrt{2}\ \pi^{-3/2}\right]\left(\frac{\pi}{8\gamma}\right)^{3/2}\frac{12\alpha+4\gamma}{9\alpha+4\gamma}\ \left(\frac{36\alpha\gamma}{9\alpha+4\gamma}\right)^{3/2}\\ &=\ -i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\pi^{-3/4}\cdot\left(\frac{3R_A^2R_M}{3R_A^2+R_M^2}\right)^{3/2}\ \frac{4R_A^2+R_M^2}{3R_A^2+R_M^2}.\end{aligned} \tag{E5}$$

Similarly, from (D7) we get

$$\begin{aligned}I_m^{(L=1)}(\mathbf{k}^2=0)\ &\stackrel{\lambda\to\infty}{\longrightarrow}\ -\frac{3}{4\pi}\ \frac{R_M}{\sqrt{2}}\ (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n^*)\ 2\sqrt{2}\ \left(\frac{\pi}{R_M^2}\right)^{3/4}\cdot\left[-\frac{1}{R_M^2}\ \frac{4\gamma}{9\alpha+4\gamma}\ \pi^{-3/2}\right]\left(\frac{36\alpha\gamma}{9\alpha+4\gamma}\right)^{3/2}\\ &=\ +\frac{3}{4\pi}\ (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n^*)\ \cdot\frac{1}{2}\sqrt{2}\ \pi^{-3/4}\cdot\left(\frac{3R_A^2R_M}{3R_A^2+R_M^2}\right)^{3/2}\ \frac{R_M}{3R_A^2+R_M^2}.\end{aligned} \tag{E6}$$

2. Overlaps for (56) → (70) Transitions

From (D9) we get

$$\begin{aligned}I_m^{(L=0)\prime\prime}(\mathbf{k}^2=0)\ &\stackrel{\lambda\to\infty}{\longrightarrow}\ +i\sqrt{\frac{3}{4\pi}}\ (\boldsymbol{\epsilon}_m\cdot\mathbf{k})\cdot\frac{9}{2}\pi^{-3/4}\cdot\left(\frac{R_A^2R_M}{3R_A^2+R_M^2}\right)^{3/2}\cdot\\ &\times\left[\left(\frac{R_A^2}{3R_A^2+R_M^2}\right)^2-\frac{R_A^2+R_M^2}{3R_A^2+R_M^2}\right].\end{aligned} \tag{E7}$$

From (D11) we get

$$\begin{aligned}I_m^{(L=1)\prime\prime}(\mathbf{k}^2=0)\ &\stackrel{\lambda\to\infty}{\longrightarrow}\ -\frac{3}{4\pi}\ (\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n^*)\ \cdot\frac{1}{4}\sqrt{6}\ \pi^{-3/4}\cdot\left(\frac{3R_A^2R_M}{3R_A^2+R_M^2}\right)^{3/2}\cdot\\ &\times\frac{R_M}{3R_A^2+R_M^2}\left(\frac{2R_A^2-R_M^2}{3R_A^2+R_M^2}\right).\end{aligned} \tag{E8}$$

3. Ratio's for Overlaps

From formulas (D12) and (D13) the limits $\lim_{\lambda\to\infty}$ give the ratio's:

(a) Ratio's for $\underline{L_M = 0}$:

$$\frac{I_m^{(L=0)\prime\prime}(\mathbf{k}^2=0)}{I_m^{(L=0)}(\mathbf{k}^2=0)} = -\frac{1}{2}\sqrt{3}\ \left[\left(\frac{R_A^2}{3R_A^2+R_M^2}\right)^2 - \frac{R_A^2+R_M^2}{4R_A^2+R_M^2}\right]. \tag{E9}$$

(b) Ratio's for $\underline{L_M = 1}$:

$$\frac{I_m^{(L=1)\prime\prime}(\mathbf{k}^2=0)}{I_m^{(L=1)}(\mathbf{k}^2=0)} = -\frac{1}{2}\sqrt{3}\ \left(\frac{2R_A^2-R_M^2}{3R_A^2+R_M^2}\right). \tag{E10}$$

4. Comparison with formulas Le Youanc et al

1. Comparison Pair-creation constants: Writing Eq. (3.12) in Ref. [3] as

$$I_m(N';N,M) \equiv -i\sqrt{\frac{3}{4\pi}}(\boldsymbol{\epsilon}_m\cdot\mathbf{k})\ \bar{I}(N';N,M), \tag{E11}$$

where

$$\bar{I}(N';N,M) = \pi^{-3/4}\left(\frac{3R_A^2R_M}{3R_A^2+R_M^2}\right)^{3/2}\ \frac{4R_A^2+R_M^2}{3R_A^2+R_M^2}. \tag{E12}$$

Then, comparing the expression for g_V in Eq. (3.13) of Ref. [3]

$$\begin{aligned} f_{NN\rho} &= \gamma\left(\frac{2}{3\pi}\right)^{1/2}\pi^{3/4}m_V^{3/2}\left(\frac{3R_A^2R_M}{3R_A^2+R_M^2}\right)^{3/2}\ \frac{4R_A^2+R_M^2}{3R_A^2+R_M^2} \\ &= \gamma\left(\frac{2}{3\pi}\right)^{1/2}\pi^{3/2}m_V^{3/2}\ \bar{I}(N';N,M), \end{aligned} \tag{E13}$$

with Eq. (50)

$$\begin{aligned} g_V &= (3/\sqrt{2})\ \pi^{-3/4}\bar{\gamma}_{q\bar{q}}\ \left(\frac{\pi}{4\lambda_0}\right)^{3/2}\frac{\sqrt{m_VR_M}}{(\Lambda_{QPC}R_M)^2} \\ &\Rightarrow \frac{3}{4}\pi^{-3/2}\bar{\gamma}_{q\bar{q}}\ \left(\frac{\pi}{4\lambda_0}\right)^{3/2}\frac{\sqrt{m_V}}{\Lambda_{QPC}^2}\cdot\bar{I}(N';N,M). \end{aligned} \tag{E14}$$

This gives

$$\bar{\gamma}_{q\bar{q}}\ \lambda_0^{-3/2} = \sqrt{\frac{2}{3}}\frac{32\pi}{3}\ \left(m_V\Lambda_{QPC}^2\right)\cdot\gamma,$$

and therefore, requiring that $\bar{\gamma}_{q\bar{q}} = \gamma$, we obtain the relation

$$\lambda_0^{-3/2} = \sqrt{\frac{2}{3}}\,\frac{32\pi}{3}\,\left(m_V\,\Lambda_{QPC}^2\right) \equiv \Lambda_0^3. \qquad \text{(E15)}$$

From (E15) we get $\Lambda_0 \approx 1$ GeV, giving $\lambda_0 \approx 0.2$ fm, which is in reasonable agreement with the actually used λ-value for the QQG form-factor.

2. Comparison Integral Formulas:

a. (56) → (56) Transitions:

a1. Without Quark-Form Factor $F(\mathbf{Q}^2) = 1$:

$$I_m^{(L=0)}(\mathbf{k}^2=0) = -i\sqrt{\frac{3}{4\pi}}\,(\boldsymbol{\epsilon}_m\cdot\mathbf{k}) \times \begin{cases} A:\ +\pi^{-3/4}\left[\left(\frac{3R_A^2 R_M}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\frac{4R_A^2+R_M^2}{3R_A^2+R_M^2}\right] \\ \to 2\sqrt{2}\,(\pi/R_M^2)^{3/4}\left[\left(\frac{3R_A^2}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\frac{4R_A^2+R_M^2}{3R_A^2+R_M^2}\right]\cdot\left(\frac{R_M}{R_\Lambda}\right)^3 \\ B:\ +\frac{7}{6}\ 2\sqrt{2}\,(\pi/R_M^2)^{3/4} \end{cases}$$

$$I_{m,n}^{(L=1)}(\mathbf{k}^2=0) = -\frac{3}{4\pi}\,(\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n^*) \times \begin{cases} A:\ -\frac{1}{2}\sqrt{2}\,\pi^{-3/4}\left[\left(\frac{3R_A^2 R_M}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\frac{R_M}{3R_A^2+R_M^2}\right] \\ \to -2R_M^{-1}\ (\pi/R_M^2)^{3/4}\ \left[\left(\frac{3R_A^2}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\ \frac{R_M^2}{3R_A^2+R_M^2}\right]\cdot\left(\frac{R_M}{R_\Lambda}\right)^3 \\ B:\ +4R_M^{-1}\ (\pi/R_M^2)^{3/4}\,. \end{cases}$$

(E16)

a2. With Quark-Form Factor $F(\mathbf{Q}^2) \neq 1$:

$$I_m^{(L=0)}(\mathbf{k}^2=0) = -i\sqrt{\frac{3}{4\pi}}\,(\boldsymbol{\epsilon}_m\cdot\mathbf{k}) \times \begin{cases} A:\ +\pi^{-3/4}\left[\left(\frac{3R_A^2 R_M}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\frac{4R_A^2+R_M^2}{3R_A^2+R_M^2}\right] \\ \to 2\sqrt{2}\,(\pi/R_M^2)^{3/4}\left[\left(\frac{3R_A^2}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\frac{4R_A^2+R_M^2}{3R_A^2+R_M^2}\right]\cdot\left(\frac{R_M}{R_\Lambda}\right)^3 \\ B:\ +2\sqrt{2}\ (\pi/R_M^2)^{3/4}\left(\frac{3R_A^2R_M^2}{3R_A^2(R_M^2+R_\Lambda^2)+R_M^2R_\Lambda^2}\right)^{3/2}\cdot \\ \qquad \times\left[CD/6+1\right], \end{cases}$$

$$I_{m,n}^{(L=1)}(\mathbf{k}^2=0) = -\frac{3}{4\pi}\,(\boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_n^*) \times \begin{cases} A:\ -\frac{1}{2}\sqrt{2}\,\pi^{-3/4}\left[\left(\frac{3R_A^2 R_M}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\ \frac{R_M}{3R_A^2+R_M^2}\right] \\ \to -2R_M^{-1}\ (\pi/R_M^2)^{3/4}\ \left[\left(\frac{3R_A^2}{3R_A^2+R_M^2}\right)^{\frac{3}{2}}\ \frac{R_M^2}{3R_A^2+R_M^2}\right]\cdot\left(\frac{R_M}{R_\Lambda}\right)^3 \\ B:\ +2R_M^{-1}\ (\pi/R_M^2)^{3/4}\left(\frac{3R_A^2R_M^2}{3R_A^2(R_M^2+R_\Lambda^2)+R_M^2R_\Lambda^2}\right)^{3/2}\cdot \\ \qquad \times\frac{R_M^2}{R_M^2+R_\Lambda^2}\left\{1-C\frac{R_\Lambda^2}{3R_A^2}\right\}. \end{cases}$$

(E17)

We note that because of (E2) for a proper comparison we have to multiply the A-entries with a factor $(\pi/4\lambda_0)^{3/2}$. Taking $\lambda_0 = R_\Lambda^2/8$, we get the expressions after $\rightarrow$ in the equations above.

Noticeable is the sign change between the A- and the B-expressions in (E16) and (E17). This is due to a sign flip in the following factor in the formulas (D9) and (D11) between $\lambda = 0$ and $\lambda \rightarrow \infty$:

$$\Big\{\ldots\Big\} = \frac{\gamma}{\gamma+\lambda} - \left[1 - \left(\frac{\gamma}{\gamma+\lambda}\right)^2\right]\left(\frac{4\bar{\alpha}\gamma}{9\alpha^2}\right)$$
$$= \frac{\gamma}{\gamma+\lambda}\left[1 - 4\lambda\frac{2\gamma+\lambda}{9\alpha(\gamma+\lambda)+4\gamma\lambda}\right],$$

which is > 0 for small λ, and < 0 for large λ.

b. (56) $\rightarrow$ (70) Transitions:

$$\frac{I_m^{(L=0)\prime\prime}(\mathbf{k}^2=0)}{I_m^{(L=0)}(\mathbf{k}^2=0)} = \begin{cases} A:\ -\frac{1}{2}\sqrt{3}\left[\left(\frac{R_A^2}{3R_A^2+R_M^2}\right)^2 - \frac{R_A^2+R_M^2}{3R_A^2+R_M^2}\right] \\ B:+\frac{1}{2}\sqrt{3}\left\{1 - C\frac{1+CD/18}{1+CD/6}\right\}, \end{cases}$$

$$\frac{I_m^{(L=1)\prime\prime}(\mathbf{k}^2=0)}{I_m^{(L=1)}(\mathbf{k}^2=0)} = \begin{cases} A:\ -\frac{1}{2}\sqrt{3}\left(\frac{2R_A^2-R_M^2}{3R_A^2+R_M^2}\right) \\ B:\ -\frac{1}{2}\sqrt{3}\left\{(1-C) - (1-5C/3)CE\cdot\frac{R_\Lambda^2}{3R_A^2}\right\}\cdot \\ \times\left(1 - \frac{R_\Lambda^2}{3R_A^2}CE\right)^{-1}. \end{cases}$$

Here,

$$C = \frac{\bar{\alpha}}{\alpha} = \frac{3R_A^2\,(R_M^2+R_\Lambda^2)}{3R_A^2\,(R_M^2+R_\Lambda^2)+R_M^2R_\Lambda^2} = 1 - \frac{R_\Lambda^2}{R_M^2}\ldots \approx 1,$$
$$D = \frac{R_M^2+2R_\Lambda^2}{R_M^2+R_\Lambda^2} = 1 + \frac{R_\Lambda^2}{R_M^2}\ldots \approx 1,\ E = \frac{2R_M^2+R_\Lambda^2}{R_M^2+R_\Lambda^2} = 3 - D.$$

Here, (A) refers to [3], and (B) to the formulas in these notes.

APPENDIX F: VECTOR DOMINANCE AND CFI

For a formulation of the basic relations of vector-dominance model (VDM) consider the hahronic reaction $VA \rightarrow B$, where V is a vector meson and A and B are arbitrary hadrons. From the LSZ-reduction formulas we have for the amplitude

$$T_\mu(VA \rightarrow B) = \langle B|j_{V,\mu}(0)|A\rangle\ , \tag{F1}$$

where $j_{V,\mu}(x)$ is the hadronic current to which the vector-meson field $V_\mu(x)$ couples, i.e.

$$\left(\partial^\rho\partial_\rho + m_V^2\right)\ V_\mu(x) = j_{V,\mu}(x)\ . \tag{F2}$$

Here, the vector current is assumed to be conserved, i.e. $\partial^\mu j_{V,\mu}(x) = 0$. Taking matrix elements of (F2) between arbitrary states, and using translational invariance, one easily derives

$$\langle B|V_\mu(0)|A\rangle = (m_V^2 - k^2)^{-1}\langle B|j_{V,\mu}(0)|A\rangle \ , \tag{F3}$$

which using CFI (se F7)

$$j_\mu^{em}(x) = \sum_{V=\rho,\omega,\phi} e\ \frac{m_V^2}{2\gamma_V} V_\mu(x)\ . \tag{F4}$$

This gives for the matrix elements of the electromagnetic current the expression

$$\langle B|j_\mu^{em}(0)|A\rangle = \sum_{V=\rho,\omega,\phi} \frac{e}{2\gamma_V}\frac{m_V^2}{m_V^2 - k^2}\langle B|j_{V,\mu}(0)|A\rangle\ . \tag{F5}$$

From the QCD electromagnetic current operator in (u,d,s)-space

$$\begin{aligned} j_\mu^{em} &= \frac{e}{3}\left[2\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d - \bar{s}\gamma_\mu s\right] \\ &= e\left[\frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right) + \frac{1}{6}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) - \frac{1}{3}\bar{s}\gamma_\mu s\right] \\ &= \frac{e}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right) + \frac{1}{3\sqrt{2}}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) - \frac{\sqrt{2}}{3}\bar{s}\gamma_\mu s\right] \\ &\equiv \sum_{V=\rho^0,\omega,\phi} e\ \frac{m_V^2}{2\gamma_V} V_\mu(x)\ . \end{aligned} \tag{F6}$$

and (F5) for $k^2 = 0$ we have

$$\begin{aligned} \frac{m_\rho^2}{\sqrt{2}\gamma_\rho}\rho_\mu(x) &= \frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right)\ , \\ \frac{m_\omega^2}{\sqrt{2}\gamma_\omega}\omega_\mu(x) &= \frac{1}{6}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right)\ , \\ \frac{m_\phi^2}{\sqrt{2}\gamma_\phi}\phi_\mu(x) &= -\frac{1}{3}\left(\bar{s}\gamma_\mu s\right)\ . \end{aligned} \tag{F7}$$

From $SU(3)$ one has

$$\begin{aligned} \rho_\mu^0 &\sim \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right)\ , \\ \phi_\mu^8 &\sim \frac{1}{\sqrt{6}}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\bar{s}\gamma_\mu s\right)\ , \\ \phi_\mu^1 &\sim \frac{1}{\sqrt{3}}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s\right)\ . \end{aligned} \tag{F8}$$

Ideal mixing gives for the physical ω and ϕ

$$\begin{aligned} \omega &= \sqrt{2/3}\ \phi^1 + \sqrt{1/3}\ \phi^8\ , \\ \phi &= -\sqrt{1/3}\ \phi^1 + \sqrt{2/3}\ \phi^8\ , \end{aligned} \tag{F9}$$

which implies that

$$\omega_\mu \;\sim\; \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) \;\;,\;\; \phi_\mu \sim -\left(\bar{s}\gamma_\mu s\right) \;. \tag{F10}$$

Comparison with (F7) implies that

$$\gamma_\rho^{-1} : \gamma_\omega^{-1} : \gamma_\phi^{-1} = 3 : 1 : \sqrt{2} \;. \tag{F11}$$

The photon vector-meson couplings have been determined in the Orsay e^+e^--experiments [35], yielding

$$\gamma_\rho^2/4\pi = 0.58 \pm 0.06 \;\;,\;\; \gamma_\omega^2/4\pi = 4.6 \pm 0.45 \;\;,\;\; \gamma_\phi^2/4\pi = 3.6 \pm 0.3 \;. \tag{F12}$$

Considering the nucleon interaction Hamiltonians

$$\mathcal{H}_I \;=\; g_{NN\rho}\bar{\psi}\gamma_\mu \boldsymbol{\tau}\psi \cdot \boldsymbol{\rho}^\mu + g_{NN\omega}\bar{\psi}\gamma_\mu\psi\omega^\mu + g_{NN\phi}\bar{\psi}\gamma_\mu\psi\phi^\mu \;, \tag{F13}$$

one finds from (F5) for the electromagnetic charge form factors of the nucleon at $k^2 = 0$, i.e. VDM hypothesis,

$$\begin{aligned} \frac{1}{2} = F_1^V(0) \;&=\; \frac{g_{NN\rho}}{2\gamma_\rho} \;, \\ \frac{1}{2} = F_1^S(0) \;&=\; \frac{g_{NN\omega}}{2\gamma_\omega} + \frac{g_{NN\phi}}{2\gamma_\phi} \;. \end{aligned} \tag{F14}$$

Assuming $g_{NN\phi} = 0$, in view of the fact that ϕ contains only s-quarks, see (F10), and the nucluon none, we get

$$g_{NN\rho} \;=\; \gamma_\rho \;\;,\;\; g_{NN\omega} = \gamma_\omega \;, \tag{F15}$$

which implies the prediction [36]

$$\frac{g_{NN\omega}}{g_{NN\rho}} = \frac{\gamma_\omega}{\gamma_\rho} = 3 \;. \tag{F16}$$

In order to make contact with the f_V-constants in [37], notice that in terms of the f_V's, equation (F7) reads

$$\begin{aligned} \frac{m_\rho^2}{\sqrt{2}f_\rho}\rho_\mu(x) \;&=\; \frac{1}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right) \;, \\ \frac{m_\omega^2}{\sqrt{2}f_\omega}\omega_\mu(x) \;&=\; \frac{1}{2}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) \;, \\ \frac{m_\phi^2}{\sqrt{2}f_\phi}\phi_\mu(x) \;&=\; \left(\bar{s}\gamma_\mu s\right) \;. \end{aligned} \tag{F17}$$

Comparison (F7) and (F17) gives

$$f_\rho = \gamma_\rho \;,\;\; f_\omega = \gamma_\omega/3 \;,\;\; f_\phi = -\gamma_\phi/3 \;. \tag{F18}$$

From (F16) and (F18) we obtain that

$$f_\rho = f_\omega = \gamma_\omega/3 \;, \tag{F19}$$

and moreover $m_\rho \approx m_\omega$. So, the ρ and ω are pretty degenerate.

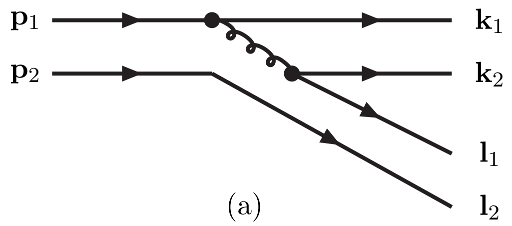

FIG. 2: $^3P_0$-*model with scalar-exchange (a).*

APPENDIX G: VPP-DECAY AND QPC

To check the relation between the pair-creation constant $\gamma_{q\bar{q}}$ in these notes and γ in the literature [3, 5], we compute the matrix element for $V \to P + P$. This process in the QPC-model is depicted in Fig. 2. There is in addition a diagram where the exchange is between the $\mathbf{p}_2$-line and the pair (b), which can be taken into account by giving the contribution from Fig. 2 an extra factor 2.

The decay matrix element is

$$\begin{aligned}
&\langle P_B(\mathbf{k})\ P_C(\mathbf{l})|S|V_A(\mathbf{p})\rangle = -i\int d^4x\ \langle P_B(\mathbf{k})\ P_C(\mathbf{l})|\mathcal{H}_I^{(S)}|V_A(\mathbf{p})\rangle = \\
&-i\frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}} \sum_{r_As_A}\sum_{r_Bs_B}\sum_{r_Cs_C}\int d^3p_1\ d^3p_2 \int d^3k_1\ d^3k_2 \int d^3l_1\ d^3l_2\ \cdot \\
&\times\delta(\mathbf{p}-\mathbf{p_1}-\mathbf{p_2})\ \delta(\mathbf{k}-\mathbf{k_1}-\mathbf{k_2})\ \delta(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\ \delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{l_1}-\mathbf{p_1})\ \delta(\mathbf{l_2}-\mathbf{p_2})\cdot \\
&\times\widetilde{\psi}_B^{(L=0)*}(\mathbf{k}_1,\mathbf{k}_2)\ \widetilde{\psi}_C^{(L=0)*}(\mathbf{l}_1,\mathbf{l}_2)\ \widetilde{\psi}_A^{(L=0)}(\mathbf{p}_1,\mathbf{p}_2)\cdot\chi^{(0)}(r_B,s_B)\ \chi^{(0)}(r_C,s_C)\ \chi_m^{(1)}(r_A,s_A)\cdot \\
&\times\langle 0|d(k_2,s_B)b(k_1,r_B)\ d(l_2,s_C)b(l_1,r_C)\sum_{i,j}\bar{q}_i(x)q_i(x)\cdot\bar{q}_j(x)q_j(x)\cdot b^\dagger(p_1,r_A)b^\dagger(p_2,s_B)|0\rangle.
\end{aligned} \tag{G1}$$

The evaluation of the vacuum expectation gives the factors

$$\begin{aligned}
\bar{u}(\mathbf{k}_1,r_B)\ u(\mathbf{p}_1,r_A)\ &\approx\ \delta_{r_A,r_B}, \\
\bar{u}(\mathbf{l}_1,r_C)\ u(\mathbf{k}_2,s_B)\ &\approx\ \frac{1}{2m_Q}(-)^{1/2-s_B}\ \left[\chi^\dagger_{-s_B}\boldsymbol{\sigma}_j\cdot(\mathbf{k}_2-\mathbf{l}_1)\chi_{r_C}\right],
\end{aligned} \tag{G2}$$

which leads to the spin factor

$$\begin{aligned}
&\delta_{r_A,r_B}\delta_{s_A,s_C}(-)^{1/2-s_B}\ \left[\chi^\dagger_{-s_B}\boldsymbol{\sigma}_j\cdot(\mathbf{k}_2-\mathbf{l}_1)\chi_{r_C}\right]\cdot \\
&\times\chi^{(0)}(r_B,s_B)\ \chi^{(0)}(r_C,s_C)\ \chi_m^{(1)}(r_A,s_A) = \\
&\Rightarrow -\frac{1}{2}\boldsymbol{\epsilon}_{-m}\cdot(\mathbf{k}_2-\mathbf{l}_1) = \frac{1}{2}\sqrt{\frac{4\pi}{3}}\mathcal{Y}_1^{-m}(\mathbf{k}_2-\mathbf{l}_1).
\end{aligned} \tag{G3}$$

The matrix element (G1) becomes

$$\langle P_B(\mathbf{k})\ P_C(\mathbf{l})|S|V_A(\mathbf{p})\rangle = -2\pi\ i\delta(E_A-E_B-E_C)(2\pi)^{-3}\ \frac{\gamma_{q\bar{q}}}{\Lambda^2_{QPC}m_Q}\cdot$$

$$\times \int d^3p_1\ d^3p_2 \int d^3k_1\ d^3k_2 \int d^3l_1\ d^3l_2\ \delta(\mathbf{p}-\mathbf{p_1}-\mathbf{p_2})\cdot$$
$$\times\delta(\mathbf{k}-\mathbf{k_1}-\mathbf{k_2})\ \delta(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\ \delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{l_1}-\mathbf{p_1})\ \delta(\mathbf{l_2}-\mathbf{p_2})\cdot$$
$$\times\widetilde{\psi}_B^{(L=0)*}(\mathbf{k}_1,\mathbf{k}_2)\ \widetilde{\psi}_C^{(L=0)*}(\mathbf{l}_1,\mathbf{l}_2)\ \widetilde{\psi}_A^{(L=0)}(\mathbf{p}_1,\mathbf{p}_2)\cdot[\boldsymbol{\epsilon}_{-m}\cdot(\mathbf{k}_2-\mathbf{l}_1)]\,. \tag{G4}$$

1. Momentum integral for $V\to PP$ (I)

We define the basic overlap integral $I_0(A;B,M)$, wher A and B are mesons, as

$$I_0(A;B,M)\equiv \int d^3p_1\ d^3p_2 \int d^3k_1\ d^3k_2 \int d^3l_1\ d^3l_2\ \delta(\mathbf{p}-\mathbf{p_1}-\mathbf{p_2})\cdot$$
$$\times\delta(\mathbf{k}-\mathbf{k_1}-\mathbf{k_2})\ \delta(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\ \delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{l_1}-\mathbf{p_1})\ \delta(\mathbf{l_2}-\mathbf{p_2})$$
$$\times\widetilde{\psi}_B^{*}(\mathbf{k}_1,\mathbf{k}_2)\ \widetilde{\psi}_C^{*}(\mathbf{l}_1,\mathbf{l}_2)\ \widetilde{\psi}_A^{(}\mathbf{p}_1,\mathbf{p}_2)$$
$$=\mathcal{N}_0 \int d^3p_1\ d^3p_2 \int d^3k_1\ d^3k_2 \int d^3l_1\ d^3l_2\ \delta(\mathbf{p}-\mathbf{p_1}-\mathbf{p_2})\cdot$$
$$\times\delta(\mathbf{k}-\mathbf{k_1}-\mathbf{k_2})\ \delta(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\ \delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{l_1}-\mathbf{p_1})\ \delta(\mathbf{l_2}-\mathbf{p_2})$$
$$\times\exp\left[+f(\mathbf{p}_1,\mathbf{p_2};\mathbf{k}_1,\mathbf{k}_2,\mathbf{l}_1,\mathbf{l}_2\right]\ , \tag{G5}$$

where in the last expression $\mathbf{k}_5$ is understood to be fixed according to the δ-function in the first expression, and in this section

$$\mathcal{N}_0=\left(\frac{R_A^2R_B^2R_C^2}{\pi^3}\right)^{3/4}\,, \tag{G6}$$

and

$$f(\mathbf{p}_1,\ldots,\mathbf{l}_2)\ =\ -\frac{1}{8}\left[R_A^2\boldsymbol{\Delta}_A^2+R_B^2\boldsymbol{\Delta}_B^2+R_C^2\boldsymbol{\Delta}_C^2\right], \tag{G7}$$

where

$$\boldsymbol{\Delta}_A\ =\ \mathbf{p_1}-\mathbf{p_2}\ ,\ \boldsymbol{\Delta}_B=\mathbf{k_1}-\mathbf{k_2}\ ,\ \boldsymbol{\Delta}_C=\mathbf{l_1}-\mathbf{l_2}. \tag{G8}$$

We work in the rest system of A, i.e. $\mathbf{p}=0$. Then, defining $\mathbf{Q}=\mathbf{p_1}-\mathbf{k_1}=\mathbf{k_2}+\mathbf{l_1}$, i.e. the momentum transfer from the meson-quarks to the created pair, we can solve for the quark momenta:

$$\mathbf{p}_1=+\frac{1}{2}\boldsymbol{\Delta}_A\ \ ,\ \mathbf{p}_2=-\frac{1}{2}\boldsymbol{\Delta}_A,$$
$$\mathbf{k}_1=+\frac{1}{2}\boldsymbol{\Delta}_A-\mathbf{Q}\ \ ,\ \mathbf{k}_2=+\frac{1}{2}\boldsymbol{\Delta}_A-\boldsymbol{\Delta}_C+\mathbf{Q},$$
$$\mathbf{l}_1=-\frac{1}{2}\boldsymbol{\Delta}_A+\boldsymbol{\Delta}_C\ \ ,\ \mathbf{l}_2=-\frac{1}{2}\boldsymbol{\Delta}_A. \tag{G9}$$

Then, the relations between the momenta from the δ-functions in (G1) are

$$\mathbf{k_1}+\mathbf{k}_2=\boldsymbol{\Delta}_A-\boldsymbol{\Delta}_C=\mathbf{k}\ ,\ \text{or}\ \boldsymbol{\Delta}_C=\boldsymbol{\Delta}_A-\mathbf{k},$$
$$\mathbf{k_1}-\mathbf{k}_2=\boldsymbol{\Delta}_B=\boldsymbol{\Delta}_A-2\mathbf{Q}-\mathbf{k},$$
$$\mathbf{l_1}+\mathbf{l}_2=-\boldsymbol{\Delta}_A+\boldsymbol{\Delta}_C=\mathbf{l}=-\mathbf{k}. \tag{G10}$$

Choosing as independent integration variables $\mathbf{Q}$ and $\mathbf{\Delta}_A$ the f-function reads

$$f(\ldots) \Rightarrow -\frac{1}{8}\left[(R_B^2+R_C^2)\mathbf{k}^2\right] - \frac{1}{8}\left[(R_A^2+R_B^2+R_C^2)+\mathbf{\Delta}^2 \right.$$
$$\left. -\,2(R_B^2+R_C^2)\ \mathbf{\Delta}_A\cdot\mathbf{k} + 4R_B^2\ \mathbf{Q}^2 + 4R_B^2\ \mathbf{Q}\cdot\mathbf{k} - 4R_B^2\ \mathbf{\Delta}_A\cdot\mathbf{Q}\right]. \qquad \text{(G11)}$$

Then, the basic integral becomes

$$\begin{aligned} I_0(A;B,M) \ =\ & \mathcal{N}_0 \exp\left\{-\frac{1}{8}(R_B^2+R_C^2)\mathbf{k}^2\right\}\cdot\left\{\frac{1}{8}\int d^3\Delta_A \int d^3Q\cdot\right. \\ & \times \exp\left(-\frac{1}{8}\left[(R_A^2+R_B^2+R_C^2)\ \mathbf{\Delta}^2 - 2(R_B^2+R_C^2)\ \mathbf{\Delta}_A\cdot\mathbf{a}\right.\right. \\ & \left.\left.\left. +\,4R_B^2\ \mathbf{Q}^2 + 4R_B^2\ \mathbf{Q}\cdot\mathbf{b} - 4R_B^2\ \mathbf{\Delta}_A\cdot\mathbf{Q}\right]\right)\right\}, \end{aligned} \qquad \text{(G12)}$$

where $\mathbf{a}=\mathbf{b}=\mathbf{k}$. Performance of the integrations yields

$$\begin{aligned} I_0(A;B,M) \quad =\quad & \mathcal{N}_0\left(\frac{2\pi}{R_B^2}\right)^{3/2}\left(\frac{2\pi}{R_A^2+R_C^2}\right)^{3/2}\ \exp\left\{-\frac{1}{8}(R_B^2+R_C^2)\mathbf{k}^2\right\}\cdot \\ & \times\exp\left\{-\frac{1}{8}\frac{\left[(R_B^2+R_C^2)\mathbf{a}-R_B^2\mathbf{b}\right]^2}{(R_A^2+R_C^2)}\right\} \\ \overset{\mathbf{a}=\mathbf{b}=\mathbf{k}}{\longrightarrow}\ & \mathcal{N}_0\left(\frac{2\pi}{R_B^2}\right)^{3/2}\left(\frac{2\pi}{R_A^2+R_C^2}\right)^{3/2}\ \exp\left[-\frac{1}{8}\frac{R_A^2(R_B^2+R_C^2)+R_B^2R_C^2}{R_A^2+R_C^2}\ \mathbf{k}^2\right]. \end{aligned} \qquad \text{(G13)}$$

We note that extra momentum in integrand:

$$\begin{aligned} \Delta_{A,i} \ &\rightarrow\ \frac{4}{R_B^2+R_C^2}\nabla_{a,i} \rightarrow -\frac{R_C^2}{R_A^2+R_C^2}\ \mathbf{k}, \\ \mathbf{Q}_i \ &\rightarrow\ -\frac{2}{R_B^2}\ \nabla_{b,i} \rightarrow -\frac{1}{2}\frac{R_C^2}{R_A^2+R_C^2}\ \mathbf{k}. \end{aligned}$$

Inclusion of factor

$$[\boldsymbol{\epsilon}_{-m}\cdot(\mathbf{k}_2-\mathbf{l}_1)] = [\boldsymbol{\epsilon}_{-m}\cdot(-\mathbf{\Delta}_A+\mathbf{Q}+2\mathbf{k})] \qquad \text{(G14)}$$

gives

$$\begin{aligned} I(A;B,M) \ =\ & \mathcal{N}_0\left(\frac{2\pi}{R_B^2}\right)^{3/2}\left(\frac{2\pi}{R_A^2+R_C^2}\right)^{3/2}\ \frac{4R_A^2+5R_C^2}{2(R_A^2+R_C^2)}\cdot \\ & \times\exp\left[-\frac{1}{8}\frac{R_A^2(R_B^2+R_C^2)+R_B^2R_C^2}{R_A^2+R_C^2}\ \mathbf{k}^2\right]\ (\boldsymbol{\epsilon}\cdot\mathbf{k}). \end{aligned} \qquad \text{(G15)}$$

2. Momentum integral for $V \to PP$ (II)

Here we evaluate the momemtum integral as appears in [3, 5]. Then, we have an extra $\delta(\mathbf{Q})$ in the integrand. The integral becomes

$$I_0'(A;B,M) = \mathcal{N}_0 \exp\left\{-\frac{1}{8}(R_B^2+R_C^2)\mathbf{k}^2\right\} \cdot \left\{\frac{1}{8}\int d^3\Delta_A \cdot \right. \\ \left. \times \exp\left(-\frac{1}{8}\left[(R_A^2+R_B^2+R_C^2)\ \boldsymbol{\Delta}^2 - 2(R_B^2+R_C^2)\ \boldsymbol{\Delta}_A\cdot\mathbf{a}\right]\right)\right\}, \tag{G16}$$

giving

$$\begin{aligned} I_0'(A;B,M) \ &= \ \mathcal{N}_0 \left(\frac{2\pi}{R_A^2+R_B^2+R_C^2}\right)^{3/2} \ \exp\left\{-\frac{1}{8}(R_B^2+R_C^2)\mathbf{k}^2\right\} \cdot \\ &\quad \times \exp\left[\frac{1}{8}\frac{(R_B^2+R_C^2)^2}{R_A^2+R_B^2+R_C^2}\ \mathbf{a}^2\right] \\ &\xrightarrow{\mathbf{a}=\mathbf{k}} \left(\frac{2\pi}{R_A^2+R_B^2+R_C^2}\right)^{3/2} \ \exp\left[-\frac{1}{8}\frac{R_A^2(R_B^2+R_C^2)}{R_A^2+R_B^2+R_C^2}\ \mathbf{k}^2\right]. \end{aligned} \tag{G17}$$

We note that extra momentum in the integrand:

$$\Delta_{A,i} \to \frac{4}{R_B^2+R_C^2}\ \nabla_{a,i}\ \to \frac{R_B^2+R_C^2}{R_A^2+R_B^2+R_C^2}\ \mathbf{k}\ , \tag{G18}$$

Inclusion of factor

$$[\boldsymbol{\epsilon}_{-m}\cdot(\mathbf{k}_2-\mathbf{l}_1)] = [\boldsymbol{\epsilon}_{-m}\cdot(-\boldsymbol{\Delta}_A+\mathbf{Q}+2\mathbf{k})] \xrightarrow{\mathbf{Q}=0} \boldsymbol{\epsilon}_{-m}\cdot(-\boldsymbol{\Delta}_A+2\mathbf{k}),$$

gives

$$\begin{aligned} I'(A;B,M) \ &= \ \mathcal{N}_0 \left(\frac{2\pi}{R_A^2+R_B^2+R_C^2}\right)^{3/2} \ \frac{2R_A^2+R_B^2+R_C^2}{R_A^2+R_B^2+R_C^2} \cdot \\ &\quad \times \exp\left[-\frac{1}{8}\frac{R_A^2(R_B^2+R_C^2)}{R_A^2+R_B^2+R_C^2}\ \mathbf{k}^2\right]\ (\boldsymbol{\epsilon}_{-m}\cdot\mathbf{k}). \end{aligned} \tag{G19}$$

3. Decay $\rho(760) \to \pi+\pi$

The phenomenological hadronic interaction Hamiltonian for the decay $\rho \to \pi+\pi$ is

$$\mathcal{H}_I = -f_{\rho\pi\pi}\ \boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}\cdot\boldsymbol{\rho}^\mu. \tag{G20}$$

In the rest frame of the ρ-meson matrix element is

$$\langle k_1,k_2|S|k\rangle = -(2\pi)^4\delta(k-k_1-k_2)\cdot(2\pi)^{-9/2}\frac{1}{8E_\rho E_1 E_2}(k_1-k_2)\cdot\rho(k) \\ \Rightarrow 2\pi\delta(m_\rho-2E_\pi)\cdot(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\ \frac{1}{2m_\rho}\frac{2}{\sqrt{2m_\rho}}\ (\mathbf{k}_\pi\cdot\boldsymbol{\rho}). \tag{G21}$$

I: The decay coupling becomes

$$
\begin{aligned}
f_{\rho\pi\pi} &= (2\pi)^{3/2} \cdot (2\pi)^{-3} \cdot \sqrt{2}\, m_\rho^{3/2} \cdot 2 \frac{\gamma_{q\bar{q}}}{\Lambda_{QPC}^2 m_Q} \left(\frac{R_A^2 R_B^2 R_C^2}{\pi^3} \right)^{3/4} \\
&\times \left(\frac{2\pi}{R_B^2} \right)^{3/2} \left(\frac{2\pi}{R_A^2 + R_C^2} \right)^{3/2} \frac{(4R_A^2 + 5R_C^2)}{2(R_A^2 + R_C^2)} \\
&= 4\gamma_{q\bar{q}}\, \pi^{-3/4} \frac{(m_\rho R_A)^{3/2}}{(\Lambda_{QPC} R_A)^2} \frac{2}{m_\rho R_A} \left(\frac{R_A R_C}{R_A^2 + R_C^2} \right)^{3/2} \frac{(4R_A^2 + 5R_C^2)}{(R_A^2 + R_C^2)}
\end{aligned}
\tag{G22}
$$

Here, we used $m_Q \approx m_\rho/2$. With $R_A = R_\rho$ and $R_B = R_C = R_\pi$ we have

$$
f_{\rho\pi\pi} = 4\gamma_{q\bar{q}}\, \pi^{-3/4} \frac{(m_\rho R_\rho)^{3/2}}{(\Lambda_{QPC} R_\rho)^2} \frac{2}{m_\rho R_\rho} \left(\frac{R_\rho R_\pi}{R_\rho^2 + R_\pi^2} \right)^{3/2} \frac{(4R_\rho^2 + 5R_\pi^2)}{(R_\rho^2 + R_\pi^2)}
\tag{G23}
$$

II: In [3] the decay coupling becomes

$$
f_{\rho\pi\pi} = \gamma \left(\frac{2}{3\pi} \right)^{1/2} \pi^{3/4} (m_\rho R_\rho)^{3/2} \left(\frac{3R_\pi^2}{2R_\pi^2 + R_\rho^2} \right)^{3/2} \left(\frac{R_\pi^2 + R_\rho^2}{2R_\pi^2 + R_\rho^2} \right)^{3/2} .
\tag{G24}
$$

From (G23) and (G24) we finally obtain the ratio

$$
\begin{aligned}
\gamma_{q\bar{q}}/\gamma &= \frac{\pi}{8}\sqrt{\frac{3}{2}}\, (\Lambda_{QPC} R_\rho)^2\, (m_\rho R_\rho) \left[\frac{R_\pi^2 + R_\rho^2}{R_\pi R_\rho} \frac{3R_\pi^2}{2R_\pi^2 + R_\rho^2} \frac{R_\pi^2 + R_\rho^2}{2R_\pi^2 + R_\rho^2} \right]^{3/2} . \\
&\times \frac{(R_\rho^2 + R_\pi^2)}{(4R_\rho^2 + 5R_\pi^2)} \equiv \frac{\pi}{8}\sqrt{\frac{3}{2}}\, (\Lambda_{QPC} R_\rho)^2\, (m_\rho R_\rho)\; F(R_\pi^2/R_\rho^2) \\
&\Rightarrow \frac{\pi\sqrt{2}}{27} (\Lambda_{QPC} R_\rho)^2\, (m_\rho R_\rho) \,, \ \text{for } \mathrm{R}_\pi = \mathrm{R}_\rho .
\end{aligned}
\tag{G25}
$$

Here we defined

$$
F(x) = 3\sqrt{3}\, x^{3/4}\, \frac{1+x}{4+5x} \left(\frac{1+x}{1+2x} \right)^3 .
$$

For $x = 1/2$ and $x = 1$ one has $F = 0.69$ and $F = 0.34$ respectively. In Table II we list the $R_\rho = R_\pi$ values for which the ratio $\gamma_{q\bar{q}}/\gamma = 1$. From this table the choice, see Table V, $\Lambda_{q\bar{q}} = 600$ MeV and $R_\rho = 0.66$ corresponds indeed to $\gamma_{q\bar{q}} \approx \gamma = 2.19$.

---

[1] L. Micu, Nucl. Phys. **B10** , 521 (1969); R. Carlitz and M. Kislinger, Phys. Rev. **D 2**, 336(1970).

[2] L.B. Okun, *Leptons and Quarks*, chapter 29, North-Holland Publishing Company 1984.

[3] A. Le Yaouanc, L. Oliver, O. Péne, and J.-C. Raynal, Phys. Rev. **D 8**, 2322 (1973); **9**, 1415 (1974).

TABLE II: Pair-creation constant $\gamma_{q\bar{q}} = \gamma$ as a function of $R_\rho = R_\pi$.

Λ_{QPC} [MeV]	$R_\rho(x=1)$	$R_\rho(x=1/2)$
250	0.99	1.25
300	0.88	1.11
400	0.72	0.91
500	0.62	0.78
600	0.55	0.69
700	0.50	0.63
800	0.46	0.58
900	0.42	0.53

[4] A. Le Yaouanc, L. Oliver, O. Péne, and J.-C. Raynal, Phys. Rev. **D 11**, 1272 (1975).

[5] M. Chaichian and R. Kögerler, Ann. Phys. **124**, 61 (1980).

[6] A. De Rújula, H. Georgi, and S.L. Glashow, Phys. Rev. **D 12**, 147 (1975).

[7] E.M. Henley and Z.-Y. Zhang, Nucl. Phys. **A472**, 759 (1987). **D 12**, 147 (1975).

[8] R. Van Royen and V.F. Weisskopf, Nuovo Cimento **50 A**, 617 91967).
This representation is the equal-time Bethe-Salpeter wave function [10]:

$$f_{\mathbf{k},\alpha}(x,y) \equiv \langle 0|T\left[q_i(x)q_j(y)\right]|M(\mathbf{k},\alpha\rangle \overset{x^0=y^0}{\longrightarrow} (0|q_i(\mathbf{x})q_j(\mathbf{y})|M(\mathbf{k},\alpha)\,,$$

using the definition $\theta[0] = 1/2$.

[9] The factor i is included in the definition of the $d^\dagger_{M,P}(\mathbf{k})$-operator. This in order to have under time-reversal $\mathcal{T}|\pi_0(\mathbf{k})\rangle = |\pi_0(-\mathbf{k})\rangle$. The reason is that under time-reversal the spin-components change sign, which implies for the spin-singlet $\chi^{(0)}(-r,-s) = -\chi^{(0)}(r,s)$ etc.

[10] M. Gell-Mann and F. Low, Phys. Rev. **51**, 350 (1951).

[11] J.D. Bjorken and S.D. Drell, *I. Relativistic Quantum Mechanics* and *II. Relativistic Quantum Fields*, McGraw-Hill Publishing Company 1965.

[12] P.A. Carruthers, *Spin and Isospin in Particle Physics*, Gordon and Breach 1971. Notice the different phase convention for the antiquark bispinor as in [11].

[13] In this footnote we recall the spherical form of the vector cross product. We have for the relation between the cartesian and sperical components

$$a_{+1} = -\frac{1}{\sqrt{2}}\left(a_x + i a_y\right)\ ,\ a_{-1} = +\frac{1}{\sqrt{2}}\left(a_x - i a_y\right)\ ,\ a_0 = a_z.$$

For the cross products one has the relation

$$\begin{aligned} -i\sqrt{2}(\mathbf{a}\times\mathbf{b})_{+1} &= -\frac{1}{\sqrt{2}}\left[(\mathbf{a}\times\mathbf{b})_x + i(\mathbf{a}\times\mathbf{b})_y\right], \\ -i\sqrt{2}(\mathbf{a}\times\mathbf{b})_{-1} &= +\frac{1}{\sqrt{2}}\left[(\mathbf{a}\times\mathbf{b})_x - i(\mathbf{a}\times\mathbf{b})_y\right], \\ -i\sqrt{2}(\mathbf{a}\times\mathbf{b})_0 &= (\mathbf{a}\times\mathbf{b})_z. \end{aligned}$$

For the dot-product one has

$$(\mathbf{a} \cdot \mathbf{b}) \;=\; (-)^m a_m b_m = \sum_{m,n} (-)^m \delta_{m,-n} a_m b_n = \sum_{ij} \delta_{ij} a_i b_j,$$

[14] C. Itzykson and J-B. Zuber, *Quantum Field Theory*, McGraw-Hill Publishing Company 1980.

[15] *Review of Particle Physics*, Particle Data Group, Phys. Rev. **D 66**, 010001-1 (2002).

[16] L. J. A. M. Somers, *Unitary Symmetries, The Symmetric Group, and Multiquark Systems*, Ph.D. thesis University of Nijmegen, 1984.

[17] The explicit relation between our $9j$-symbols and those of [18] eq. (6.4.4) is

$$\begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} \;=\; \left[(2j_{13}+1)(2j_{31}+1)(2j_{23}+1)(2j_{32}+1)\right]^{1/2}$$

$$\times \begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix}$$

[18] A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton (1957)).

[19] E. Leader and E. Predazzi, *An introduction to gauge theories and modern particle physics*, Vol. I, chapter 12, Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology, Editors T. Ericson and P.V. Landshoff, Cambridge University Press 1996.

[20] W. Hai-Jun et al., Chinese Physics C, **33** (2009) 170.

[21] H. Crater and P. Vanalstine, Phys. Rev. **D 70**, 034026 (2004).

[22] B. Povh and J. Hufner, Phys. Lett. **B245** (1990) 653.

[23] J. Schwinger, Phys. Lett. B **24**, 473 (1967); Phys. Rev. **167**, 1432 (1968).

[24] F.E. Close, *An Introduction to Quarks and Partons*, Academic Press, 1979. See: chapter 4.

[25] N. Isgur and G. Karl, Phys. Rev. **D 21** (1980) 3175.

[26] M. Anselmino, E. Predazzi, S. Ekelin, S. Frederikson, and D.B. Lichtenberg, Rev. Mod. Phys. Vol. **65**, 1199 (1993).

[27] A. Le Yaouanc, L. Oliver, O. Péne, and J.-C. Raynal, Phys. Rev. **D 12**, 2137 (1975); **18**, 1591 (1978).

[28] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley Publishing Company, 1995.

[29] L.Ya. Glozman and D.O. Riska, Phys. Rep. **268** (1997) 263; L.Ya. Glozman et al, Nucl. Phys. **A631** (1998) 469c.

[30] M.M. Nagels, T.A. Rijken, and J.J. de Swart, Phys. Rev. D**17**, 768 (1978).

[31] J.J. Sakurai, *Invariance Principles and Elementary Particles*, Princeton University Press 1964, section 9.4.

[32] L.J. Reinders, H.R. Rubinstein, and S. Yazaki, Phys. Rep. **C 127**, 1 (1985).

[33] J.L. Powell and B. Crasemann, *Quantum Mechanics*, Second printing - January 1962, Addison-Wesley 1962.

[34] In connection with time-reversal invariance it is better to use as basis functions in configuration space the spherical harmonics with an extra i^L [18]

$$\mathcal{Y}_L^m(\theta,\varphi) = i^L \; r^L Y_L^m(\theta,\varphi). \tag{G26}$$

This implies for L=1 an extra factor i also in momentum space. This way the scalar- and axial-vector meson couplings become real.
[35] D. Benaksas et al, Phys. Lett., **33B**, 289 (1972); ibid **42B**, 507 (1972); C. Cosme et al, Phys. Lett., **48B**, 155 (1974).
[36] It is amusing to notice that from ideal mixing one has $g_{NN\phi} = 0$. This gives for the $SU(3)$-singlet coupling $g_1 = g_8/\tan\theta_V$, where $g_8 = (4\alpha - 1)g_\rho/\sqrt{3}$. Using g_1 one finds $g_{NN\omega} = g_8/\sin\theta_V = \sqrt{3}g_8 = (4\alpha - 1)g_{NN\rho} = 3g_{NN\rho}$. This is the $SU(6)$ result, as expected because this requires $\alpha = 1$.
[37] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 385 (1979); ibid 448 (1979).