Multi-Pomeron Exchange and the Universal Repulsion in Nuclear/Hyperonic Matter
Triple-, Quadruple- and N-tuple-Pomeron Vertices

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I. INTRODUCTION

In these notes we derive the effective two-body baryon-baryon (nucleon-nucleon, hyperon-nucleon, hyperon-hyperon) force in matter from the triple-, quadruple-pomeron, and more general the N-tuple-pomeron, vertex.

General motivation: It was found by Nishizaki, Takatsuka and Yamamoto [1] that the soft-core interactions tend to give a too low maximum for the neutron star mass, which is $M_{\text{max}} = 1.44M_\odot$. To remedy this they add a repulsive universal TBF. This is all the more necessary since the discovery of the two-solar mass neutron stars [2, 3].

Like in Ref. [4], we consider the three- and also the four-body interactions between baryons as generated by the triple- and quadruple-pomeron vertex (see [5, 6] for references). Then, we integrate out one or two of the baryons to give an effective two-body potential.

In this note, we consider the triple-, quadruple-, and N-tuple-body interaction between baryons as a given by the triple-, quadruple-, and N-tuple-pomeron vertex. The framework we use is the description of the Pomeron with a scalar field $\sigma_P(x)$. It is a ghost-field in the sense that the propagator is gaussian with (-)-sign. So, the Pomeron does not propagate and gives only so-called ‘contact’ interactions with a Gaussian form factor. This is the picture used in [7] and also in the spirit of the Reggeon Field-theory formalism, see e.g. [6] and references.

Remarks: (i) We give two derivations of the effective two-body potentials: with (a) Cartesian coordinates $x_i$, and with Jacobian coordinates $x_\alpha$. (ii) The multi-pomeron Lagrangians are without division by 3! and 4! for the triple- and quadruple-couplings respectively. As a consequence the effective two-body potentials get combinatorial factors 3! respectively 4!. (iii) The pomeron-vertices are defined with ‘unrationalized couplings’ $G_P, G_{3P}$, and $G_{4P}$ for the pomeron-baryon, the triple-pomeron, and quadruple-pomeron couplings respectively. The ‘rationalized couplings’ are defined as $g_P = G/\sqrt{4\pi}, g_{3P} = G_{3P}/(\sqrt{4\pi})^{3/2}$, and $g_{4P} = G_{4P}/(4\pi)^2$.

The content of these notes is as follows. In section II we review the two-body potential from pomeron-exchange. In section III the three-body potential is given and the effective two-body is derived, using in configuration space simple cartesian vectors for the position of the baryons. Similarly, in section IV and section V this is done for the four-body and N-body potentials. In section VI we discuss the triple- and quadruple couplings in connection with the Regge field-theory perspective. In Appendix A the derivation of the configuration space potentials is reviewed, within the context of the used normalization of the non-relativistic one-particle states. In Appendix B the three-body configuration-space potentials are derived using Jacobian coordinates for the baryons. Similarly in Appendix C for the four-body potentials. Finally, in Appendix D the Jacobian coordinates are described in more detail.

Combinatorial factors: Associate $\sigma(x)$ with the Pomeron, and the BBP coupling

$$\mathcal{L}_{BBP} = g_P \left[ \bar{\psi}(x)\psi(x) \right] \sigma(x). \quad (1.1)$$

The triple and quartic pomeron self-interactions we define as

$$\mathcal{L}_{PPP} = g_{3P} \sigma^3(x), \quad \mathcal{L}_{PPP} = g_{4P} \sigma^4(x). \quad (1.2)$$

a. Triple-pomeron exchange three-body force: 4th order diagram

$$\mathcal{M}_{3P} \sim \frac{1}{4!} \left[ \mathcal{L}_{3P} + \mathcal{L}_{BBP} \right]^4 \Rightarrow 4 \times \frac{1}{4!} \mathcal{L}_{3P} \mathcal{L}_{BBP}^3$$

$$\rightarrow \text{Combinatorial factor diagram : } 4 \times \frac{1}{4!} \times 3! = 1.$$

b. Quartic-pomeron exchange four-body force: 5th order diagram

$$\mathcal{M}_{4P} \sim \frac{1}{5!} \left[ \mathcal{L}_{3P} + \mathcal{L}_{BBP} \right]^5 \Rightarrow 5 \times \frac{1}{5!} \mathcal{L}_{4P} \mathcal{L}_{BBP}^4$$

$$\rightarrow \text{Combinatorial factor diagram : } 5 \times \frac{1}{5!} \times 4! = 1.$$

Conclusion: The Lagrangians in (1.2) give no extra combinatorial factors in the 3- and 4-body potential diagram.

II. TWO-BODY POTENTIAL FROM POMERON-EXCHANGE

Because of the universal coupling strength of the Pomeron to Baryons, we can restrict ourselves to nucleons, without loss of generality. We start from the pomeron-interaction.
The corresponding three-body potential in configuration space is given by

\[ V_M(x_1, x_2, x_3) = \prod_{i=1}^3 \left( \int \frac{d^3 p_i'}{(2\pi)^3} \frac{d^3 p_i}{(2\pi)^3} e^{-i(p'_i \cdot x_i' - p_i \cdot x_i)} M_{3P}(p'_1, p'_2, p'_3; p_1, p_2, p_3) \right) \delta \left( \sum p'_i - \sum p_i \right). \]  

III. THREE-BODY POTENTIAL FROM THE TRIPLE-POMERON VERTEX

For the triple-pomeron vertex we take the Lagrangian

\[ \mathcal{L}_{PP} = G_{3P} \mathcal{M} \sigma_{P}^3(x) \]  

Then, the matrix element for the graph of 1 is given by

\[ M_{3P}(p'_1, p'_2, p'_3; p_1, p_2, p_3) = G_{3P} G_M^3 \mathcal{M} \prod_{i=1}^3 \left\{ [\bar{u}(p'_i)u(p_i)] \Delta^P_M[(p'_i - p_i)^2] \right\} \approx G_{3P} G_M^3 \mathcal{M} \prod_{i=1}^3 \Delta^P_M[(p'_i - p_i)^2]. \]

The corresponding three-body potential in configuration space is given by

\[ q_i = \frac{1}{2} (p'_i + p_i), \quad k_i = p'_i - p_i, \]  

\[ p'_i = q_i + \frac{1}{2} k_i, \quad p_i = q_i - \frac{1}{2} k_i. \]  

For the volume integral we get

\[ I^{(2)}_V = \int d^3 r_{12} V_P(r_{12}) = G_{3P}^2 / \mathcal{M}^2. \]
Then, we have that \( d^3 p_i/d^3 p_i = d^3 q_i/d^3 q_i \), and
\[
p'_i \cdot x'_i - p_i \cdot x_i = q_i \cdot (x'_i - x) + \frac{1}{2} k_i \cdot (x'_i + x_i)
\] (3.5)

The \( q_i \)-integrations can be done immediately,
\[
\int d^3 q_i \exp \{q_i \cdot (x'_i - x_i)} = (2\pi)^3 \delta(x'_i - x_i).
\]

After this we get for the three-body potential
\[
V(x'_1, x'_2, x'_3; x_1, x_2, x_3) \equiv V(x_1, x_2, x_3) \delta(x'_1 - x_1) \delta(x'_2 - x_2) \delta(x'_3 - x_3)
\] (3.6a)
\[
V(x_1, x_2, x_3) = G_{3P} G_P^3 \left[ \prod_{i=1}^{3} \int \frac{d^3 k_i}{(2\pi)^3} \exp \left( \frac{1}{i} k_i \cdot x_i \delta(k_i + k_2 + k_3) \right) \right. \\
\left. \times \exp \left( -k_1^2/4m_P^2 \right) \exp \left( -k_2^2/4m_P^2 \right) \exp \left( -k_3^2/4m_P^2 \right) \right]
\] (3.6b)

where the Pomeron propagator \( \Delta_P^P(k^2) \) given in Eq. (2.1b) is used.

A. The triple pomeron effective two-body potential

To obtain the effective two-body potential in a baryonic medium, we integrate over the coordinate \( x_3 \) of the third nucleon. From (3.6b) it is evident that this gives a factor \((2\pi)^3 \delta(k_3)\). Using this we get from (3.6b) the two-body potential
\[
V_{eff}(x_1, x_2) = \rho_{NM} \int d^3 x_3 \left. V(x_1, x_2, x_3) \right|
\] (3.7a)
\[
V_{eff}(x_1, x_2) = G_{3P} G_P^3 \rho_{NM} \left[ \frac{2}{M^5} \right. \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \exp \left( -\frac{1}{i} k_1 \cdot x_1 \right) \exp \left( -\frac{1}{i} k_2 \cdot x_2 \right) \right. \\
\left. \times \exp \left( -k_1^2/4m_P^2 \right) \exp \left( -k_2^2/4m_P^2 \right) \exp \left( -k_3^2/4m_P^2 \right) \right]
\] (3.7b)

In the last expression we introduced the rationalized couplings
\[
g_P = G_P/\sqrt{4\pi} \, , \, g_{3P} = G_{3P}/(4\pi)^{3/2}.
\] (3.8)

Note that
(i) \( g_P \) is the Pomeron parameter in the Nijmegen potential program and papers.
(ii) result (3.7b) should be multiplied by the combinatorial factor: \( 3! \)
From (3.7b) one sees that if $g_{3P} > 0$ this gives repulsion in a few/many-body system.

Comparing formula (3.7b) with formula (8.3) in the ESC08c paper [10]

$$V_{eff}(x_1, x_2) = g_{3P}^3 \rho_{NM} \frac{1}{4\pi} \frac{4}{\sqrt{\pi}} \left( \frac{m_P}{\sqrt{2}} \right)^3 \exp \left( -\frac{1}{2} m_P^2 r_{12}^2 \right),$$

shows that $g_{3P}' = 8g_{3P}$.

Now, one has that

$$\rho_{NM} = \frac{2p_F^3}{3\pi^2}, \quad \rho_0 = \frac{p_F^3}{6\pi^2}, \quad \rho_{NM} = 4\rho_0.$$  

The volume integral of $V_{eff}$ is

$$I_{V_{eff}} = g_{3P}^3 \rho_{NM} \frac{1}{4\pi} \frac{4}{\sqrt{\pi}} = g_{3P}^3 \frac{2}{3\pi^2} \left( \frac{p_F}{\mathcal{M}} \right)^3 \cdot \frac{1}{\mathcal{M}^2} \cdot \frac{4}{\sqrt{\pi}}$$  

IV. FOUR-BODY POTENTIAL FROM THE QUADRUPLE-POMERON VERTEX

For the quadruple-pomeron vertex we take the Lagrangian

$$\mathcal{L}_{4P} = G_{4P} \sigma_{4P}(x)$$  

Then, the matrix element for the graph of 1 is given by

$$M_{4P}(p'_1, p'_2, p'_3, p'_4; p_1, p_2, p_3, p_4) = G_{4P} G_{4P}^\dagger \prod_{i=1}^4 \left\{ \frac{\bar{u}(p'_i) u(p_i)}{M} \Delta_F(p'_i - p_i)^2 \right\} \approx G_{4P} G_{4P}^\dagger \prod_{i=1}^4 \Delta_F(p'_i - p_i)^2.$$  

The corresponding four-body potential in configuration space is given by

$$V(x'_1, x'_2, x'_3, x'_4, x_1, x_2, x_3, x_4) = \prod_{i=1}^4 \left[ \int \frac{d^3p'_i}{(2\pi)^3} \int \frac{d^3p_i}{(2\pi)^3} e^{-i(p'_i \cdot x'_i - p_i \cdot x_i)} \right] \cdot \times M_{4P}(p'_1, p'_2, p'_3, p'_4; p_1, p_2, p_3, p_4) \delta \left( \sum_{i=1}^4 p'_i - \sum_{i=1}^4 p_i \right).$$  

Introducing now the combinations

$$q_i = \frac{1}{2} (p'_i + p_i), \quad k_i = p'_i - p_i, \quad \text{or}$$

$$p'_i = q_i + \frac{1}{2} k_i, \quad p_i = q_i - \frac{1}{2} k_i.$$  

Then, we have that $d^3p'_i d^3p_i = d^3q_i d^3k_i$, and

$$p'_i \cdot x'_i - p_i \cdot x_i = q_i \cdot (x'_i - x_i) + \frac{1}{2} k_i \cdot (x'_i + x_i).$$
Again, the \( q_i \)-integrations can be done immediately, leading to the four-body potential

\[
V(x'_1, x'_2, x'_3, x'_4; x_1, x_2, x_3, x_4) \equiv V(x_1, x_2, x_3, x_4) \Pi_{i=1}^4 \delta(x'_i - x_i),
\]

(4.6a)

\[
V(x_1, x_2, x_3, x_4) = G_{4P}G_P^4 \Pi_{i=1}^4 \int \frac{d^3k_i}{(2\pi)^3} e^{-ik_i \cdot x_i} \cdot \delta(k_1 + k_2 + k_3 + k_4) \cdot \\
\times \exp(-k_1^2/4m_P^2) \exp(-k_2^2/4m_P^2) \exp(-k_3^2/4m_P^2) \exp(-k_4^2/4m_P^2) \cdot \mathcal{M}^{-8},
\]

(4.6b)

A. The quadruple effective two-body potential

To obtain the effective two-body potential in a baryonic medium, we integrate over the coordinate \( x_3 \) and \( x_4 \) of the third and fourth nucleon. From (4.6b) it is evident that this gives the factors \((2\pi)^3\delta(k_3)\) and \((2\pi)^3\delta(k_4)\). Using this we get from (4.6b the two-body potential

\[
V_{\text{eff}}(x_1, x_2) = \rho_{NM}^2 \int d^3x_3 \int d^3x_4 V(x_1, x_2, x_3, x_4)
\]

\[
V_{\text{eff}}(x_1, x_2) = G_{4P}G_P^4 \rho_{NM}^2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{-ik_1 \cdot x_1} e^{-ik_2 \cdot x_2} \cdot \\
\times \delta(k_1 + k_2) \exp(-k_1^2/4m_P^2) \exp(-k_2^2/4m_P^2) \exp(-k_3^2/2m_P^2)
\]

(4.7a)

\[
= G_{4P}G_P^4 \rho_{NM}^2 \frac{4}{M^8} \frac{4}{\sqrt{\pi}} \frac{(m_P/\sqrt{2})^3}{\sqrt{\pi}} \exp\left(-\frac{1}{2} m_P^2 r_{12}^2\right).
\]

Again, we introduced in the last line the rationalized 4-point coupling \( g_{4P} = G_{4P}/(4\pi)^2 \), similar to the rationalized 3-point coupling \( g_{3P} \).

Note that the result (4.7a) should be multiplied by the combinatorial factor 4! From (4.7a) it follows that if \( g_{4P} > 0 \) this gives repulsion in a few/many-body system.

Now, one has that

\[
\rho_{NM} = \frac{2p_F^3}{3\pi^2}, \quad \rho_0 = \frac{p_F^3}{6\pi^2}, \quad \rho_{NM} = 4\rho_0.
\]

(4.8)

The volume integral of \( V_{\text{eff}} \) is

\[
I_{V_{\text{eff}}} = g_{4P}g_P^4 \rho_{NM}^2 = g_{4P}g_P^4 \frac{4}{9\pi^4} \left(\frac{p_F}{\mathcal{M}}\right)^6 \cdot \frac{1}{M^2}.
\]

(4.9)

V. N-BODY POTENTIAL FROM THE N-TUPLE-POMERON VERTEX

The work of the foregoing sections is easily generalized to the case of an N-tuple-pomeron vertex. For the N-tuple-pomeron vertex we take the Lagrangian

\[
\mathcal{L}_N = G_P^{(N)} M^{4-N} \sigma_P^N(x)
\]

(5.1)
The N-body potential is

\[
V(x'_1, \ldots, x'_N; x_1, \ldots, x_N) \equiv V(x_1, \ldots, x_N) \Pi_{i=1}^N \delta(x'_i - x_i),
\]

\[
V(x_1, \ldots, x_N) = G_P^{(N)} G_P \Pi_{i=1}^N \left\{ \int \frac{d^3k_i}{(2\pi)^3} e^{-ik_i \cdot x_i} \right\} \cdot \delta(k_1 + k_2 + \ldots + k_N) \cdot \exp(-k_i^2/4m_P^2) \ldots \exp(-k_N^2/4m_P^2) \cdot M^{4-3N},
\]

(5.2a)

Similarly to the section III the effective two-body potential in a baryonic medium is obtained by integrating over the coordinates \(x_3, \ldots, x_N\) of the nucleons (baryons). From (5.2b) it is evident that this gives the factors \((2\pi)^3\delta(k_3)\ldots(2\pi)^3\delta(k_4)\). Using this we get from (5.2b) the two-body potential

\[
V_{eff}^{(N)}(x_1, x_2) = \rho_{NM}^{N-2} \int d^3x_3 \ldots \int d^3x_N \cdot \int d^3k_1 \cdot \delta(k_1 + k_2) \exp(-k_i^2/4m_P^2) \cdot \exp(-k_1^2/2m_P^2) \cdot M^{(N-4)/2} \cdot g_P^{[N-2]} g_P^{N-2} \cdot \frac{8}{\pi \sqrt{\pi}} \cdot \left(\frac{m_P}{\sqrt{2}}\right)^3 \cdot \exp\left(-\frac{1}{2} \cdot m_P^2 r_{12}^2\right).
\]

Using this we get from (5.2b) the two-body potential

\[
V_{eff}^{(N)}(x_1, x_2) = G_P^{(N)} G_P \cdot \frac{\rho_{NM}^{N-2}}{M^{3N-4}} \cdot \int d^3k_1 \cdot \exp(-k_1^2/2m_P^2) \cdot \exp(-k_1^2/2m_P^2) \cdot \exp(-k_1^2/2m_P^2) \cdot \exp(-k_1^2/2m_P^2)
\]

(5.3)

Therefore, if \(g_P^{(N)} > 0\) this gives repulsion in a few/many-body system. In (5.3) we introduced the rationalized coupling \(G_P^{(N)} = G_P^{(N)}/(4\pi)\).

VI. DISCUSSION AND CONCLUSION

The relation between the triple and quadruple couplings and the Regge residues is as follows:

(i) Triple-pomeron coupling: The relation between the pomeron coupling \(g_P\) and the residue of the pomeron is given by [7]

\[
G_P^2 = \gamma_0^2(0) \left(\frac{s}{M^2}\right)^{\alpha_P(0)},
\]

where \(s \approx (6 - 8)M^2\). Analogously, the relation between the triple-pomeron coupling \(g_{3P}\) and the triple-residue is given by

\[
G_{3P} = r_0(0) \left(\frac{s}{M^2}\right)^{3\alpha_P(0)/2}.
\]

Therefore,

\[
\frac{G_{3P}}{G_P} = \frac{r_0(0)}{\gamma_0(0)} \left(\frac{s}{M^2}\right)^{\alpha_P(0)} \approx (6 - 8) \cdot \frac{r_0(0)}{\gamma_0(0)}.
\]
According to [5] \( r_0(0)/\gamma_0(0) = 1/40 \) and therefore we expect \( G_{3P}/G_P \approx (0.15 - 0.20) \). Comparing this with the result of the previous section implies that what is needed in the nuclear saturation is a factor two larger as expected from the triple-pomeron contribution. This leaves room for a contribution also from the change in the vector- (and scalar-) meson masses, which we used in [12].

(ii) Quadruple-pomeron coupling:
Similarly to the triple-pomeron vertex, taking the relation between the quadruple-pomeron coupling \( g_{4P} \) and the quadruple-residue \( q_0 \) as given by

\[
G_{4P} = q_0(0) \left( \frac{\bar{s}}{M^2} \right)^{2\alpha_P(0)} .
\]

Then,

\[
\frac{G_{4P}}{G_P} = \frac{q_0(0)}{\gamma_0(0)} \left( \frac{\bar{s}}{M^2} \right)^{3\alpha_P(0)/2} \approx (14.5 - 22.5) \frac{q_0(0)}{\gamma_0(0)} .
\]

(iii) Quadruple-pomeron in Reggeon field theory:
In Reggeon field theory, see e.g. [6], the (bare) gap \( \Delta_0 \) of the pomeron intercept i.e. \( \alpha_P(0) = 1 - \Delta_0 \) and the (bare) triple- and quartic- couplings, respectively \( r_0 \) and \( \lambda_0 \), is related by \( \Delta_0 = -r_0^2/\lambda_0 \). For an estimate we identify: \( g_{3P} = r_0 \) and \( g_{4P} = 4\lambda_0 \). In comparing with Regge phenomenology of the total cross sections we do not distinguish here between 'bare' and 'renormalized' quantities. In fitting the high-energy pp cross sections, Donnachie and Landshoff [13] used the 'hard' and the 'soft' pomeron trajectories \( \alpha_0(t) \) and \( \alpha_1(t) \) respectively:

\[
\alpha_0(t) = 1 - \Delta_0 + \alpha't, \quad \alpha_1(t) = 1 - \Delta_1 + \alpha't, 
\]

For the soft pomeron they fitted \( \Delta_1 = -0.0667 \), and for the hard pomeron \( \Delta_0 = -0.452 \). Using the soft pomeron and the relation above from [6], we find

\[
G_{4P} = -4r_0^2/\Delta_1 \approx 60G_{3P}^2,
\]

which gives \( G_{4P}/4\pi \approx 30 \) for \( G_{3P}/4\pi = 0.2 \). So, apart from the precise numbers for the parameters the result seems to be that \( G_{4P} >> G_{3P} \).

**Remark:** Also \( G_{3P} \) and \( G_{4P} \) are running coupling constants. Therefore for low energies these couplings may be larger than in the Regge-regime.

(iv) Polynomial-pomeron coupling:
Consider a general polynomial pomeron-vertex, using the Lagrangian

\[
L_{Pol.} = \sum_{N=3}^{\infty} G_{P}^{(N)} M^{4-N} \sigma_P^N(x).
\]

Then, from the results above the effective two-body repulsion is given by

\[
V_{\text{eff}}^{(Pol)}(x_1, x_2) = \sum_{N=3}^{\infty} \left[ \frac{g_P^{(N)} g_P^{(N)} \rho_{N}^{N-2}}{M^{4-N}} \right] \frac{1}{4\pi} \frac{4}{\sqrt{\pi}} \left( \frac{m_P}{\sqrt{2}} \right)^3 \exp \left(-\frac{1}{2}m_P^2t_{12}^2\right)
\equiv \frac{1}{4\pi} \frac{4}{\sqrt{\pi}} \left( \frac{m_P}{\sqrt{2}} \right)^3 \exp \left(-\frac{1}{2}m_P^2t_{12}^2\right) \cdot f(g_P, \rho_{MN}),
\]

9
with the volume-integral

\[
I_{V,\text{eff}}^{(N)} = \sum_{N=3}^{\infty} g_P^{(N)} g_P^N \frac{\rho_{NM}^{N-2}}{M^{3N-4}} = f(g_P, \rho_{NM}).
\]  

(6.8)

FIG. 2: CM One-boson-exchange graphs: The dashed lines with momentum \( k \) refers to the bosons: pseudo-scalar, vector, axial-vector, or scalar mesons.

**APPENDIX A: DERIVATION CONFIGURATION-SPACE POTENTIALS**

In Fig. 2 the two time-ordered graphs are drawn for a scalar exchange process. In momentum space the matrix element from (a) and (b) is, realizing that two time-ordered graphs are equivalent to a single Feynman graph,

\[
\langle p'_1, p'_2 | M | p_1, p_2 \rangle = -G^2 \delta^3(p'_1 + p'_2 - p_1 - p_2) \frac{1}{\omega_k^2},
\]  

(A1)

where we used that in the CM-frame energy conservation makes the energy transfer zero, and the notation \( \omega_k = \sqrt{k^2 + m^2} \).

Splitting off the CM-motion goes as follows. With

\[
\mathbf{R} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2), \quad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2,
\]

\[
\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2), \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,
\]

the two-particle wave function is

\[
(x_1, x_2 | p_1, p_2) = \exp \left[ i(p_1 + p_2) \cdot \mathbf{R} \right] \cdot \exp \left[ \frac{i}{2}(p_1 - p_2) \cdot \mathbf{r} \right].
\]
In configuration space

\[
\langle x'_1, x'_2 | M | x_1, x_2 \rangle = \int \frac{d^3p'_1}{(2\pi)^6} \int \frac{d^3p'_2}{(2\pi)^6} \int \frac{d^3p_1}{(2\pi)^6} \int \frac{d^3p_2}{(2\pi)^6} \langle x'_1 | p'_1 \rangle \langle x'_2 | p'_2 \rangle \langle p_2 | x_2 \rangle \langle p_1 | x_1 \rangle \cdot
\]
\[
\times \langle p'_1, p'_2 | M | p_1, p_2 \rangle = (2\pi)^{-12} \int \frac{d^3p'_1}{(2\pi)^6} \int \frac{d^3p'_2}{(2\pi)^6} \int \frac{d^3p_1}{(2\pi)^6} \int \frac{d^3p_2}{(2\pi)^6} \cdot
\]
\[
\times e^{-i(p'_1 \cdot x'_1 + p'_2 \cdot x'_2)} e^{+i(p_1 \cdot x_1 + p_2 \cdot x_2)} \langle p'_1, p'_2 | M | p_1, p_2 \rangle = (2\pi)^{-12} .
\]
\[
\times \int d^3P' d^3p' \int d^3P d^3p \ e^{-i(P' \cdot R' - P \cdot R)} e^{-i(P' \cdot r' - P \cdot r)} \langle p', P' | M | p, P \rangle .
\]
(A2)

With

\[
(p', P' | M | p, P) = \delta(P' - P) M(p', p)
\]

Performing the P and P' integrations one obtains

\[
\langle x'_1, x'_2 | M | x_1, x_2 \rangle = (2\pi)^{-3} \delta(R' - R) \langle r' | M | r \rangle ,
\]
(A3a)

\[
(r' | M | r) = (2\pi)^{-6} \int \int d^3p' d^3p \ e^{-i(p' \cdot r' - P \cdot r)} M(p', p).
\]
(A3b)

Introducing the standard variables

\[
q = \frac{1}{2} (p' + p) , \; k = p' - p,
\]
(A4)

and replacing \( \int d^3q d^3p = \int d^3q d^3k \), the q integrations can be executed immediately. One gets for \( M(k) = -G^2/\omega^2(k) \)

\[
(r' | M | r) = (2\pi)^{-6} \int \int d^3q d^3k \ e^{-i(q \cdot r' - r)} e^{-i(k \cdot r' + r)/2} M(q, k)
\]
(A5a)

\[
\Rightarrow \int \frac{d^3k}{(2\pi)^3} e^{-i(k \cdot r)} M(k).
\]
(A5b)

For Pomeron exchange \(-1/\omega^2 \rightarrow + \exp(-k^2/\Lambda^2)/\mathcal{M}^2\). Then, one has with \( r_{12} = x_1 - x_2 \),

\[
\langle x'_1, x'_2 | M_P | x_1, x_2 \rangle = (2\pi)^{-3} \delta(R' - R) \langle r' | V_P | r_{12} \rangle ,
\]

\[
V_P(r_{12}) = \frac{G^2}{4\pi} \frac{1}{2\sqrt{\pi}} \frac{\Lambda^3}{\mathcal{M}^2} e^{-m_P^2 r_{12}^2} = \frac{G^2}{4\pi} \frac{4}{2\sqrt{\pi}} \frac{m_P^3}{\mathcal{M}^2} e^{-m_P^2 r_{12}^2}.
\]
(A6)

which explains Eq. 2.3.

**APPENDIX B: THREE-BODY CONFIGURATION-SPACE POTENTIALS, JACOBIAN-COORDINATES METHOD**

The free three-particle wave function is

\[
\psi_3(x_1, x_2, x_3) = \Pi_{i=1}^3 \left[ e^{ip_i \cdot x_i} \right] .
\]
(B1)
The matrix element corresponding to the triple-pomeron graph in Fig. 1 is

\[ (p'_1, p'_2, p'_3 | M | p_1, p_2, p_3) = G_3 p G_3^2 \Pi_{i=1}^3 \left[ \frac{e^{-k_i^2/\Lambda^2}}{\mathcal{M}^2} \right] \left( \sum_i p'_i - \sum_i p_i \right), \quad (B2) \]

where \( k_i = p'_i - p_i \).

The Jacobi-coordinates in configuration and momentum space are defined as

\[ x_\rho = \frac{1}{\sqrt{2}}(x_1 - x_2) \quad p_\rho = \frac{1}{\sqrt{2}}(p_1 - p_2) \quad (B3a) \]

\[ x_\lambda = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3) \quad p_\lambda = \frac{1}{\sqrt{6}}(p_1 + p_2 - 2p_3) \quad (B3b) \]

\[ R_3 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) \quad P_3 = \frac{1}{\sqrt{3}}(p_1 + p_2 + p_3). \quad (B3c) \]

One has

\[ \sum_{i=1}^3 p_i \cdot x_i = p_\rho \cdot x_\rho + p_\lambda \cdot x_\lambda + P_3 \cdot R_3, \]

\[ \sum_{i=1}^3 k_i^2 = k_\rho^2 + k_\lambda^2 + (P'_3 - P_3)^2. \]

The potential is given by

\[ (x_1, x_2, x_3 | V_3 | x_1, x_2, x_3) = \Pi_{i=1}^3 \left[ \int d^3 p'_i \int d^3 p_i \psi_3^*(x'_1, x'_2, x'_3) (p'_1, p'_2, p'_3 | M_3 p | p_1, p_2, p_3) \cdot \right. \]

\[ \times \psi_3(x_1, x_2, x_3) = (2\pi)^{-18} \int d^3 P_3 d^3 p'_3 d^3 p_\lambda d^3 P_\rho \exp \left\{ -i(P'_3 \cdot R'_3 - P_3 \cdot P_3) \right\} \]

\[ \times \exp \left\{ -i(p'_\rho \cdot x'_\rho - p_\rho \cdot x_\rho) \right\} \exp \left\{ -i(p'_\lambda \cdot x'_\lambda - p_\lambda \cdot x_\lambda) \right\} \cdot \]

\[ \times G_3 p G_3 \left[ M^2 \right]^{-\frac{3}{2}} \exp \left\{ -(k_\rho^2 + k_\lambda^2)/\Lambda \right\} \cdot \]

\[ \times \exp \left\{ -(P'_3 - P_3)^2/\Lambda^2 \right\} (3\sqrt{3})^{-1} \delta^3(P'_3 - P_3). \quad (B4) \]

Since everything factorizes we can perform all integrals in an elementary way. The integrals are

\[ I_{CM} = (2\pi)^{-3} \int d^3 P'_3 d^3 P_3 \exp \left\{ -i(P'_3 \cdot R'_3 - P_3 \cdot P_3) \right\} \exp \left\{ -(P'_3 - P_3)^2/\Lambda^2 \right\}. \]

\[ \times \delta^3(P'_3 - P_3) = \delta^3(R'_3 - R_3) \quad (B5a) \]

\[ I_\rho = (2\pi)^{-6} \int d^3 p'_\rho d^3 p_\rho \exp \left\{ -i(p'_\rho \cdot x'_\rho - p_\rho \cdot x_\rho) \right\} \exp \left\{ -k_\rho^2/\Lambda^2 \right\} \]

\[ = \delta^3(x'_\rho - x_\rho) \left( \frac{\Lambda}{2\sqrt{\pi}} \right)^3 \exp \left\{ -\frac{1}{4} \Lambda^2 x_\rho^2 \right\} \quad (B5b) \]

\[ I_\lambda = (2\pi)^{-6} \int d^3 p'_\lambda d^3 p_\lambda \exp \left\{ -i(p'_\lambda \cdot x'_\lambda - p_\lambda \cdot x_\lambda) \right\} \exp \left\{ -k_\lambda^2/\Lambda^2 \right\} \]

\[ = \delta^3(x'_\lambda - x_\lambda) \left( \frac{\Lambda}{2\sqrt{\pi}} \right)^3 \exp \left\{ -\frac{1}{4} \Lambda^2 x_\lambda^2 \right\}. \quad (B5c) \]
Separating the \( \delta \)-functions by defining
\[
(x_1, x_2, x_3 | V_3 | x_1, x_2, x_3) = \left[ \prod_{i=1}^3 \delta^3(x'_i - x_i) \right] V_3(x_1, x_2, x_3)
\]  
the potential becomes
\[
V_3(x_1, x_2, x_3) = (2\pi)^{-9} g_{3P}^3 G_{\rho \cdot MN}^3 \left( \frac{\Lambda}{M} \right)^6 \left( \frac{\pi}{\sqrt{3}} \right)^3 \exp \left[ -\frac{1}{4} \Lambda^2 (x_\rho^2 + x_\lambda^2) \right]
\]  
(Integration over particle 3 gives
\[
V_{eff}(x_1, x_2) = \rho_{MN} \int d^3 x_3 \; V(x_1, x_2, x_3).
\]
Translating the integrand back to the variables \( x_i, i = 1, 2, 3 \) we have
\[
f_3 \equiv x_\rho^2 + x_\lambda^2 = \frac{2}{3} \left( x_1^2 + x_2^2 + x_3^2 - x_1 \cdot x_2 - x_1 \cdot x_3 - x_2 \cdot x_3 \right),
\]
which leads to the \( x_3 \)-integral
\[
\int d^3 x_3 \exp \left[ -\frac{1}{6} \Lambda^2 \left( x_3^2 - x_3 \cdot (x_1 + x_2) \right) \right] = \left( \frac{6\pi}{\Lambda^2} \right)^{3/2} \exp \left[ \frac{1}{24} \Lambda^2 (x_1 + x_2)^2 \right]
\]
giving
\[
V_{eff}(x_1, x_2) = (2\pi)^{-9/2} g_{3P}^3 G_{\rho \cdot MN}^3 (2)^{-3} \left( \frac{\Lambda}{M} \right)^5 \exp \left[ -\frac{1}{8} \Lambda^2 (x_1 - x_2)^2 \right]
\]
\[
= (2\pi)^{-9/2} g_{3P}^3 G_{\rho \cdot MN}^3 \frac{m_\rho^3}{M^5} \exp \left[ -\frac{1}{2} \frac{m_\rho^2 r_{12}^2}{M} \right],
\]
where we used \( \Lambda = 2m_\rho \). Inserting the rationalized couplings \( g_P, g_{3P} \) defined by \( G_P = \sqrt{4\pi} g_P \) and \( G_{3P} = (4\pi)^{3/2} g_{3P} \) one has
\[
V_{eff}(x_1, x_2) = g_{3P} g_P^3 \frac{\rho_{MN}}{M^5} \cdot \frac{2}{\pi} \frac{4}{\sqrt{\pi}} \left( \frac{m_\rho}{\sqrt{2}} \right)^3 \exp \left[ -\frac{1}{2} \frac{m_\rho^2 r_{12}^2}{M} \right],
\]
This formula agrees with (3.7b)!

**APPENDIX C: FOUR-BODY CONFIGURATION-SPACE POTENTIALS, JACOBIAN-COORDINATES METHOD**

The free four-particle wave function is
\[
\psi_4(x_1, x_2, x_3) = \Pi_{i=1}^4 \left[ e^{ip_i \cdot x_i} \right].
\]
The matrix element corresponding to the triple-pomeron graph in Fig. 1 is
\[
(p'_1, p'_2, p'_3, p'_4 | M | p_1, p_2, p_3, p_4) = G_4 P G^4 \mathcal{M} \prod_{i=1}^4 \left[ \frac{e^{-k_i^2/\Lambda^2}}{\mathcal{M}^2} \right] \left( \sum_i p'_i - \sum_i p_i \right),
\]
where \( k_i = p'_i - p_i \).

The Jacobi-coordinates in configuration and momentum space are defined as

\[
x_\rho = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad p_\rho = \frac{1}{\sqrt{2}}(p_1 - p_2)
\]

\[
x_\lambda = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3), \quad p_\lambda = \frac{1}{\sqrt{6}}(p_1 + p_2 - 2p_3)
\]

\[
x_\mu = \frac{1}{\sqrt{12}}(x_1 + x_2 + x_3 - 3x_4), \quad p_\mu = \frac{1}{\sqrt{12}}(p_1 + p_2 + p_3 - 3p_4)
\]

\[
R_4 = \frac{1}{\sqrt{4}}(x_1 + x_2 + x_3 + x_4), \quad P_4 = \frac{1}{\sqrt{4}}(p_1 + p_2 + p_3 + p_4).
\]

One has

\[
\sum_{i=1}^{4} p_i \cdot x_i = p_\rho \cdot x_\rho + p_\lambda \cdot x_\lambda + p_\mu \cdot x_\mu + P_4 \cdot R_4,
\]

\[
\sum_{i=1}^{4} k_i^2 = k_\rho^2 + k_\lambda^2 + k_\mu^2 + (P'_4 - P_4)^2.
\]

The potential is given by

\[
(x_1', x_2', x_3', x_4'|V_4|x_1, x_2, x_3, x_4) = \Pi_{i=1}^{4} \left[ \int d^3p_i' \int d^3p_i \right] \psi_4(x_1', x_2', x_3', x_4') \cdot
\]

\[
\times (p'_1, p'_2, p'_3, p'_4|M_4|p_1, p_2, p_3, p_4) \psi_4(x_1, x_2, x_3, x_4) =
\]

\[
(2\pi)^{-24} \int d^3p_1'd^3p_2'd^3p_3'd^3p_4 \int d^3P d^3p_\rho d^3p_\lambda d^3p_\mu \exp \left[ -i(P'_4 \cdot R'_4 - P_4 \cdot P_4) \right] \cdot
\]

\[
\times \exp \left[ -i(p'_\rho \cdot x'_\rho - p_\rho \cdot x_\rho) \right] \exp \left[ -i(p'_\lambda \cdot x'_\lambda - p_\lambda \cdot x_\lambda) \right] \exp \left[ -i(p'_\mu \cdot x'_\mu - p_\mu \cdot x_\mu) \right] \cdot
\]

\[
\times G_{4p}G_{p}^4 \left[ M^2 \right]^{-4} \exp \left\{ -(k_\rho^2 + k_\lambda^2 + k_\mu^2) / \Lambda \right\} \cdot
\]

\[
\times \exp \left\{ -(P'_4 - P_4)^2 / \Lambda^2 \right\} \left( 4\sqrt{4} \right)^{-1} \delta^2(P'_4 - P_4).
\]

(C4)

Since everything factorizes we can perform all integrals in an elementary way. The integrals
are

\[ I_{CM} = (2\pi)^{-3} \int d^3P'_4 d^3P_4 \exp \left[ -i(P'_4 \cdot R'_4 - P_4 \cdot P_4) \right] \exp \left\{ -(P'_4 \cdot P_4)^2/\Lambda^2 \right\} \times \delta^3(P'_4 - P_4) = \delta^3(R'_4 - R_4) \]

(C5a)

\[ I_\rho = (2\pi)^{-6} \int d^3p'_\rho d^3p_\rho \exp \left[ -i(p'_\rho \cdot x'_\rho - p_\rho \cdot x_\rho) \right] \exp \left\{ -k^2_\rho/\Lambda^2 \right\} \]

(C5b)

\[ I_\lambda = (2\pi)^{-6} \int d^3p'_\lambda d^3p_\lambda \exp \left[ -i(p'_\lambda \cdot x'_\lambda - p_\lambda \cdot x_\lambda) \right] \exp \left\{ -k^2_\lambda/\Lambda^2 \right\} \]

(C5c)

\[ I_\mu = (2\pi)^{-6} \int d^3p'_\mu d^3p_\mu \exp \left[ -i(p'_\mu \cdot x'_\mu - p_\mu \cdot x_\mu) \right] \exp \left\{ -k^2_\mu/\Lambda^2 \right\} \]

(C5d)

Separating the \( \delta \)-functions by defining

\[ (x'_1, x'_2, x'_3, x'_4) = [\prod_{i=1}^4 \delta^3(x'_i - x_i)] \quad V_4(x_1, x_2, x_3, x_4) \]

(C6)

the potential becomes

\[ V_4(x_1, x_2, x_3, x_4) = (2\pi)^{-12} G_4 p G^4_\rho \mathcal{M} \left( \frac{\Lambda}{\mathcal{M}} \right)^9 (\pi)^{9/2} \]

\[ \times \exp \left[ -\frac{1}{4} \Lambda^2 (x^2_\rho + x^2_\lambda + x^2_\mu) \right]. \]

(C7)

Integration over particle 3 and 4 gives

\[ V_{eff}(x_1, x_2) = \rho^2_{MN} \int d^3x_3 d^3x_4 \ V(x_1, x_2, x_3, x_4). \]

(C8)

Translating the integrand back to the variables \( x_i, i = 1, 2, 3 \) we have

\[ f_4 \equiv x^2_\rho + x^2_\lambda + x^2_\mu = \frac{3}{4} (x^2_1 + x^2_2 + x^2_3 + x^2_4) \]

\[ -\frac{1}{2} (x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 x_1 \cdot x_4 + x_2 \cdot x_4 + x_3 \cdot x_1) \]

\[ = \frac{3}{4} (x^2_1 + x^2_2 - \frac{2}{3} x_1 \cdot x_2) + \frac{3}{4} (x^2_3 + x^2_4) \]

\[ -\frac{1}{2} [(x_1 + x_2) \cdot (x_3 + x_4) + x_3 \cdot x_4] \]

\[ = \frac{3}{4} (x^2_1 + x^2_2 - \frac{2}{3} x_1 \cdot x_2) + \frac{1}{2} (x_3 - x_4)^2 \]

\[ + \frac{1}{4} (x_3 + x_4)^2 - \frac{1}{2} (x_1 + x_2) \cdot (x_3 + x_4). \]
Introducing \( x = (x_3 + x_4)/2 \) and \( y = x_3 - x_4 \) leads to the 34-integrals
\[
\int d^3x d^3y \exp \left[ -\frac{1}{4}\Lambda^2 \left\{ x^2 - x \cdot (x_1 + x_2) \right\} - \frac{1}{8}\Lambda^2 y^2 \right] =
\left( \frac{8\pi}{\Lambda^2} \right)^{3/2} \left( \frac{4\pi}{\Lambda^2} \right)^{3/2} \exp \left[ \frac{1}{16}\Lambda^2 (x_1 + x_2)^2 \right]
\]
giving
\[
V_{\text{eff}}(x_1, x_2) = (2\pi)^{-9/2} G_4 P G_4 \rho_{MN} \frac{\Lambda^3}{M^8} \exp \left[ -\frac{1}{8}\Lambda^2 (x_1 - x_2)^2 \right]
= (2\pi)^{-9/2} G_4 P G_4 \rho_{MN} (2\sqrt{2}) m_P^2 \exp \left[ -\frac{1}{2} m_P^2 r_{12}^2 \right], \tag{C9}
\]
where we used \( \Lambda = 2m_P \). Inserting the rationalized couplings \( g_p, g_4P \) defined by \( G_P = \sqrt{4\pi} g_p \) and \( G_4P = (4\pi)^2 g_4P \) one has
\[
V_{\text{eff}}(x_1, x_2) = 8 g_4P g_P \rho_{MN} \frac{\rho_{MN}}{M^8} \cdot \frac{4}{\sqrt{\pi}} \left( \frac{m_P}{\sqrt{2}} \right)^3 \exp \left[ -\frac{1}{2} m_P^2 r_{12}^2 \right], \tag{C10}
\]
This formula agrees with (4.7a)!

APPENDIX D: JACOBI-COORDINATES A=4 SYSTEMS

For an N-body system the Jacobian coordinates \( r_i \) are constructed via the following rules:
\[
\begin{align*}
r_1 &= x_1 - x_2, \tag{D1a} \\
r_j &= \sum_{k=1}^{j} \frac{m_k}{m_{0j}} x_k - x_{j+1}, \quad m_{0j} = \sum_{k=1}^{j} m_k. \tag{D1b}
\end{align*}
\]

FIG. 3: Jacobi-coordinates of a four particle system.
Here, \( x_{N+1} = 0 \) and for \( j = N \) this is defined as \( r_N \equiv R \) the center of mass

\[
R = \frac{1}{M} \sum_{k=1}^{N} m_k x_k, \quad M = m_0 N = \sum_{k=1}^{N} m_k. \quad (D2)
\]

For \( N=4 \) this leads to the Jacobian coordinates

\[
\begin{align*}
 r_1 &= x_1 - x_2, \quad (D3a) \\
 r_2 &= R_{12} - x_3 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} - x_3, \quad (D3b) \\
 r_3 &= R_{123} - x_4 = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} - x_4, \quad (D3c) \\
 R &= R_{1234} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}{m_1 + m_2 + m_3 + m_4}. \quad (D3d)
\end{align*}
\]

The inverse of (D3) reads

\[
\begin{align*}
 x_1 &= R + \frac{m_2}{m_1 + m_2} r_1 + \frac{m_3}{m_1 + m_2 + m_3} r_2 + \frac{m_4}{m_1 + m_2 + m_3 + m_4} r_3, \quad (D4a) \\
 x_2 &= R - \frac{m_1}{m_1 + m_2} r_1 + \frac{m_3}{m_1 + m_2 + m_3} r_2 + \frac{m_4}{m_1 + m_2 + m_3 + m_4} r_3, \quad (D4b) \\
 x_3 &= R - \frac{m_1 + m_2}{m_1 + m_2 + m_3} r_2 + \frac{m_4}{m_1 + m_2 + m_3 + m_4} r_3, \quad (D4c) \\
 x_4 &= R - \frac{m_1 + m_2 + m_3}{m_1 + m_2 + m_3 + m_4} r_3. \quad (D4d)
\end{align*}
\]

1. Four-pomeron Potential

For the multi-pomeron potentials for the leading term we neglect the baryon mass-differences. Therefore we take \( m_1 = m_2 = m_3 = m_4 \). Then,

\[
\begin{align*}
 r_1 &= x_1 - x_2 = \sqrt{2} x_\nu, \quad (D5a) \\
 r_2 &= \frac{1}{2} (x_1 + x_2) - x_3 = \sqrt{\frac{3}{2}} x_\lambda, \quad (D5b) \\
 r_3 &= \frac{1}{3} (x_1 + x_2 + x_3) - x_4 = \sqrt{\frac{4}{3}} x_\mu, \quad (D5c) \\
 r_4 &= \frac{1}{4} (x_1 + x_2 + x_3 + x_4) = \sqrt{\frac{1}{4}} R. \quad (D5d)
\end{align*}
\]

with the inverse

\[
\begin{align*}
 x_1 &= R + \frac{1}{2} r_1 + \frac{1}{3} r_2 + \frac{1}{4} r_3, \quad (D6a) \\
 x_2 &= R - \frac{1}{2} r_1 + \frac{1}{3} r_2 + \frac{1}{4} r_3, \quad (D6b) \\
 x_3 &= R - \frac{2}{3} r_2 + \frac{1}{4} r_3, \quad (D6c) \\
 x_4 &= R - \frac{3}{4} r_3. \quad (D6d)
\end{align*}
\]
Analogous to the A=3 case we work with the configuration and momentum space Jacobi-variables

\[ x_\rho = \frac{1}{\sqrt{2}}(x_1 - x_2), \quad p_\rho = \frac{1}{\sqrt{2}}(p_1 - p_2) , \]  
\[ x_\lambda = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3), \quad p_\lambda = \frac{1}{\sqrt{6}}(p_1 + p_2 - 2p_3) , \]  
\[ x_\mu = \frac{1}{\sqrt{12}}(x_1 + x_2 + x_3 - 3x_4), \quad p_\mu = \frac{1}{\sqrt{12}}(p_1 + p_2 + p_3 - 3p_4) , \]  
\[ R = \frac{1}{4}(x_1 + x_2 + x_3 + x_4), \quad P = \frac{1}{4}(p_1 + p_2 + p_3 + p_4) . \]  

(D7a) (D7b) (D7c) (D7d)

This gives

\[ \sum_{i=1}^{4} p_i = P \cdot R + p_\rho \cdot x_\rho + p_\lambda \cdot x_\lambda + p_\mu \cdot x_\mu \]  
\[ \text{(D8)} \]

The connection with the Jacobi-coordinates used in the case of the triton is given by

\[ r_1 = \rho, \quad r_2 = \sqrt{\frac{3}{2}} \lambda , \]  
\[ \text{(D9)} \]

which indeed yields

\[ (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2 = 3(\rho^2 + \lambda^2) . \]

In Fig. 3 the constellation of the different vectors are displayed. We note that only particle 4 is connected with the center of mass.


[8] The normalization of the one-particle states is [7] \( \langle p'|p \rangle = (2\pi)^3 \delta^3(p' - p) \), and the one-particle wave function is \( \langle x|p \rangle = \exp(i p \cdot x) \). This differs a factor \((2\pi)^{3/2}\) compared to the normalization used in [9]. Important relations are

\[ \int d^3x \ |x\rangle\langle x| = 1 , \quad \int \frac{d^3p}{(2\pi)^3} \ |p\rangle\langle p| = 1 \]
and the relation of matrix elements in configuration and momentum space reads

\[(r'|V|r) = \int \int \frac{d^3p'}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} (p'|V|p)(p|r)\]

\[= \int \int \frac{d^3qd^3k}{(2\pi)^6} e^{i(q·(r'−r))} e^{i(k·(r'+r))/2} V(q,k)\]

where \(q = (p' + p)/2, k = p' - p\).


