Quark-Baryon Di-quark-exchange in SU(3)

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The quark-baryon (QB) matrix elements for di-quark exchange are derived, which have application in a medium of mixed nuclear-quark matter. The diquark-exchange for QB $\rightarrow$ BQ is of the axial-vector type. Similar as for nucleons, the potential for $Q+B \rightarrow Q+B$ is repulsive for S-waves, P-waves, and other partial waves. Parameterizing the strength of this interaction as a function of the quark density such as to increase with the deconfining rate, it can become rather significant in neutron stars. This in particular in connection with the "hyperon-puzzle" and the two-solar mass neutron stars.

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I. INTRODUCTION

At present the mixed nuclear-quark matter is a much attended topic, see [1]. In this note we derive the quark-baryon (QB) couplings due to di-quark D) exchange.

In considering meson coupling to quarks as well as to baryons leads in a natural way to a coupled channel treatment in a mixture of nuclear and quark matter. The tri-quark presentation of the nucleon [2], *i.e.* $B \sim \eta_B = (\widetilde{q} C \gamma^\mu q)\gamma_5\gamma_\mu q$ suggests the reactions $B \leftrightarrow 3Q$, which takes place in high density matter.
the treatment of the Lagrangian with the tri-quark field $\eta_B(x)$ in *e.g.* the functional form of the partition function for matter is difficult to handle. This problem is circumvented by avoiding third powers in the quark fields by the introduction of an auxiliary colored di-quark field $\chi^a_\mu(x)$ [3], which upon quantization leads to di-quarks D. Apart from this technical reason it may be that di-quark configurations play a real rôle.
In [4] nucleon and quark mixed matter is discussed in the context of the MF-approach in matter in the framework of the grand-partition functional. In [5] the di-quark propagator is derived using field-theoretical methods. In these notes we work out the baryon-couplings using SU(3)-symmetry.
Exchange of di-quarks leads in all (S-, P-, etc) waves to a repulsive interaction between the quark and a baryon. Crucial is the di-quark field $\chi^a_\mu(x)$*-propagator* $i\Delta^{ab}_{\mu\nu}(x'-x) = (0|T\left[\chi^a_\mu(x')\chi^{b\dagger}_\nu(x)\right]|0)$. *A detailed study [5] shows that compared to axial-meson-exchange there is a (-)-sign difference in the di-quark propagator coming from the Wick-expansion theorem.*
The SU(3)-extension to hyperon-quark channels $Y_{\{8\}} + Q_{\{3\}} \rightarrow Q_{\{3\}} + Y_{\{8\}}$ with di-quark-exchange $(QQ)_{\{6\}}$ leads to potentials similar in character to that for $N + Q \rightarrow Q + N$.

The contents of this paper is as follows. In section II the representation of the tri-quark states for baryons is given as well as the Lagrangian for the description of the mixed matter of quarks and baryons with the baryon tri-quark transition. Furthermore the di-quark are introduced. In section II B the SU(3) generalization to hyperons and s-quarks is given. In section A details of the evaluation of the couplings for different B+Q channels are given.

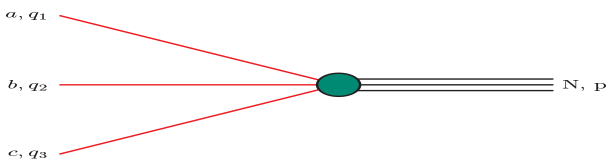

FIG. 1: Tri-quark-Nucleon Vertex

II. INTRODUCTION DI-QUARK-QUARK PRESENTATION BARYONS

In mixed quark-nucleon matter there are, depending on the densities, transitions between nucleons and tri-quarks. For the tri-quark system we choose the operator, see [2]

$$\eta_N(x) = \big(\tilde{q}^a(x) C\gamma^\mu q^b(x)\big)\gamma_5\gamma_\mu q^c(x)\varepsilon^{abc} \tag{2.1}$$

where C is the charge conjugation operator in Dirac space, which has in the PD-representation the properties $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$, $C = -C^{-1} = -C^\dagger = -C^T$. For the proton and neutron this is

$$\eta_p(x) = \big(\tilde{u}^a(x) C\gamma^\mu u^b(x)\big)\gamma_5\gamma_\mu d^c(x)\varepsilon^{abc}, \tag{2.2a}$$
$$\eta_n(x) = \big(\tilde{d}^a(x) C\gamma^\mu d^b(x)\big)\gamma_5\gamma_\mu u^c(x)\varepsilon^{abc}, \tag{2.2b}$$

where a,b,c denote the $SU_c(3)$ color indices of the quark fields.

A direct way to treat this system in *e.g.* a mean-field-theory (MFT) would be to introduce the auxiliary tri-quark field η_N via the Lagrangian density

$$\mathcal{L}_\eta \sim \bar{\eta}\eta - \left[\bar{\eta}\big(q^a(x) C\gamma^\mu q^b(x)\big)\gamma_5\gamma_\mu q^c(x)\varepsilon^{abc} + h.c.\right]$$

which via the E.L:. equations gives for the composite field $\eta = \left(\tilde{q}^a(x)C\gamma^\mu q^b(x)\right)\gamma_5\gamma_\mu q^c(x)\varepsilon^{abc}$.
However, the occurrence of a triple-quark field makes a handling of the partition function Z_G very complicated. In the tri-quark nucleon presentation (2.1) the contraction of the indices, indicated by (...), suggest to introduce instead the di-quark field.

A. Mixed matter: Mean-field with Di-quarks

For a $B \to 3Q$ interaction Lagrangian with the tri-quark field $\eta_N(x)$ the functional form of the partition function is difficult to handle. In order to avoid third powers in the quark fields we write $\eta_N(x)$ in terms of the (bosonic) di-quark field $\chi^a_\mu(x)$ as

$$\eta_N(x) = (\hbar c)^2\chi^a_\mu(x)\gamma_5\gamma^\mu q^a(x),\ \chi^a_\mu(x) \equiv \varepsilon^{abc}\tilde{q}^b(x)C\gamma_\mu q^c(x)/(\hbar c)^2. \quad (2.3)$$

Introduction this auxiliary di-quark field χ^a_μ via the Lagrangian density [3]

$$\mathcal{L}_\chi \sim \chi^{a\dagger}_\mu(x)\chi^{\mu a}(x) - \left[\chi^{a\dagger}_\mu(x)\left(\tilde{q}^b(x)C\gamma^\mu q^c(x)\right)\varepsilon^{abc} + h.c.\right] \quad (2.4)$$

gives via the E.L. equation $\chi^a_\mu(x) \sim \left(\tilde{q}^b(x)C\gamma_\mu q^c(x)\right)\varepsilon^{abc}$. For $N \leftrightarrow 3Q$ interaction $\mathcal{L}_{int}$ with di-quark takes the form

$$\mathcal{L}_{int} \to -\bar{\lambda}_3\left\{(\bar{\psi}(x)\gamma_5\gamma^\mu q^a)\ \chi^a_\mu + h.c.\right\}, \quad (2.5)$$

where $\bar{\lambda}_3 = (\hbar c)^2\lambda_3$. From now on we use the notation $\lambda_3 \equiv \bar{\lambda}_3$.

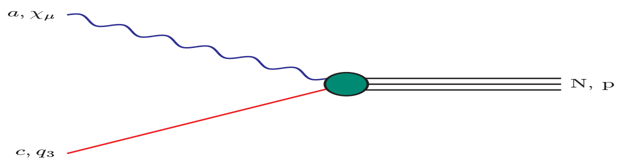

FIG. 2: Di-quark-Quark-Nucleon Vertex

B. Baryons Tri-quark and Di-quark-quark Configurations

To generalize the di-quark-approach for the SU(3) octet baryons, for the tri-quark system we choose operators $\eta_B, B = N, \Lambda, \Sigma, \Xi$, for which we can assume that there is a significant transition with the baryons *i.e.* $\langle 0|\eta_B(x)|B\rangle \neq 0$, see [2]. For $p, n, \Lambda, \Sigma, \Xi$ suitable η_B-operators are [6]

$$\eta_p(x) = \left(\tilde{u}^a(x)C\gamma^\mu u^b(x)\right)\gamma_5\gamma_\mu d^c(x)\varepsilon^{abc}, \quad (2.6a)$$
$$\eta_n(x) = \left(\tilde{d}^a(x)C\gamma^\mu d^b(x)\right)\gamma_5\gamma_\mu u^c(x)\varepsilon^{abc}, \quad (2.6b)$$
$$\eta_\Lambda(x) = \sqrt{\frac{2}{3}}\Big[\left(\tilde{u}^a(x)C\gamma^\mu s^b(x)\right)\gamma_5\gamma_\mu d^c(x) - \left(\tilde{d}^a(x)C\gamma^\mu s^b(x)\right)\gamma_5\gamma_\mu u^c(x)\Big]\varepsilon^{abc}, \quad (2.6c)$$
$$\eta_\Sigma(x) = \left(\tilde{u}^a(x)C\gamma^\mu u^b(x)\right)\gamma_5\gamma_\mu s^c(x)\varepsilon^{abc}, \quad (2.6d)$$
$$\eta_\Xi(x) = -\left(\tilde{s}^a(x)C\gamma^\mu s^b(x)\right)\gamma_5\gamma_\mu u^c(x)\varepsilon^{abc}, \quad (2.6e)$$

where a,b,c denote the $SU_c(3)$ color indices of the quark fields. C is the charge conjugation operator in Dirac space, which has in the PD-representation the properties $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$, $C = -C^{-1} = -C^\dagger = -C^T$. Note that since the two-quark systems are anti-symmetric in color they are symmetric in flavor. For the octet baryons the di-quark fields we introduce as

$$P, \Sigma^+ : \chi^a_\mu(uu) = \left(\tilde{u}^b(x)C\gamma^\mu u^c(x)\right)\varepsilon^{abc}, \quad (2.7a)$$
$$N, \Sigma^- : \chi^a_\mu(dd) = \left(\tilde{d}^b(x)C\gamma^\mu d^c(x)\right)\varepsilon^{abc}, \quad (2.7b)$$
$$\Lambda, \Sigma^0 : \chi^a_\mu(us) = \left(\tilde{u}^b(x)C\gamma^\mu s^c(x)\right)\varepsilon^{abc}, \quad (2.7c)$$
$$\Lambda, \Sigma^0 : \chi^a_\mu(ds) = \left(\tilde{d}^b(x)C\gamma^\mu s^c(x)\right)\varepsilon^{abc}, \quad (2.7d)$$
$$\Xi : \chi^a_\mu(ss) = \left(\tilde{s}^b(x)C\gamma^\mu s^c(x)\right)\varepsilon^{abc}. \quad (2.7e)$$

Fig. 3 pictures the exchange potential V_e for the baryon-quark di-quark-exchange interaction $BQ \to QB$.

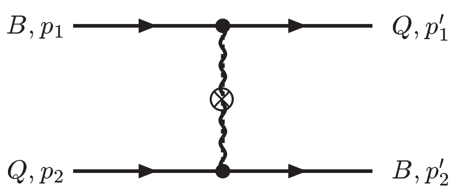

FIG. 3: Di-quark-exchange for $BQ \to QB$ reaction.

Now we generalize the $N \leftrightarrow 3Q$ interaction $\mathcal{L}_{int}$ (2.5) to the $B \leftrightarrow 3Q$ interaction

$$\mathcal{H}_{int} \to \bar{\lambda}_3(Y)\left\{(\bar{\psi}_B(x)\gamma_5\gamma^\mu q^a)\ \chi^a_\mu(B) + h.c.\right\}, \quad (2.8)$$

where $\bar{\lambda}_3(Y) = (\hbar c)^2\lambda_3(Y)$, and we used a notation for the di-quark fields appropriate for each baryon, se (2.7).

In deriving further the di-quark-exchange potentials for baryons Λ, Σ, Ξ, we note that all ingredients and steps are the same for baryons and nucleons. It turns out that all di-quark-exchange potentials $B + Q \to Q + B$ lead to repulsive QB-potentials for the baryon-octet members. This because the effective Lagrangian has the same sign for all baryons etc.
The reactions $s + \Lambda, \Sigma \to n + N$ from di-quark-exchange give transition potential having the same spin-space structure as the di-quark-exchange potentials from $n, s + N, \Lambda, \Sigma \to N, \Lambda, \Sigma + n, s$.
The dimensionless tri-quark couplings are related to the QCD Sum Rules parameter λ_N as follows, see [6] equation (4.71b),

$$\bar{\lambda}_3^2 \approx \frac{M_N^6 e}{2(2\pi)^4(\hbar c)^6} \sim 10^{-3}(M_N/\hbar c)^6,$$

giving $\bar{\lambda}_3/\sqrt{4\pi} \approx 0.96$.

III. GENERAL SU$_F$(3)-COUPLINGS $B \to D + Q$

We use the notations from Ref. [7] for the wave function, Clebsch-Gordan coefficients, and Isoscalar factors:

$$\psi\left(\begin{array}{ccc} \mu_1 & \mu_2 & \mu \\ & & \nu \end{array}\right) = \sum_{I_1,Y_1,I_2,Y_2} \left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu \\ I_1Y_1 & I_2Y_2 & IY \end{array}\right) \chi\left(\begin{array}{ccc} \mu_1 & \mu_2 & \\ I_1Y_1 & I_2Y_2 & II_zY \end{array}\right), \tag{3.1}$$

where the first factor on the r.h.s. is the so-called Isoscalar factor [7, 8], and

$$\chi\left(\begin{array}{ccc} \mu_1 & \mu_2 & \\ I_1Y_1 & I_2Y_2 & II_zY \end{array}\right) = \sum_{I_{1z},I_{2z}} C\left(\begin{array}{ccc} I_1 & I_2 & I \\ I_{1z} & I_{2z} & I_z \end{array}\right) \phi_{\nu_1}^{(\mu_1)}\phi_{\nu_2}^{(\mu_2)}, \tag{3.2}$$

which leads to the SU(3) C.G.C.

$$\left(\begin{array}{ccc} \mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu \end{array}\right) = C\left(\begin{array}{ccc} I_1 & I_2 & I \\ I_{1z} & I_{2z} & I_z \end{array}\right) \left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu \\ I_1Y_1 & I_2Y_2 & IY \end{array}\right) \tag{3.3}$$

Here, the hypercharge Y=B+S where B= baryon number and S= strangeness. This gives for quarks $Y = 1/3, 1/3, -2/3$ for u, d, s respectively. For the baryons $Y = 1, 0, 0, -1$ for N, Λ, Σ, Ξ respectively.
The Wigner-Eckart theorem for SU(3) reads [7]

$$\left(\phi_{\nu_f}^{(\mu_f)}|\mathcal{H}_{int}^{(1)}|\psi_{\nu_i}^{(\mu_i)}\right) = \left(\begin{array}{ccc} \mu_i & 1 & \mu_f \\ \nu_i & 0 & \nu_f \end{array}\right) \left(\mu_f||\mathcal{H}_{int}^{(1)}||\mu_i\right)\delta_{\mu_i,\mu_f}, \tag{3.4}$$

where $\mu_i = \mu_f = \{8\}$ and $\mathcal{L}_{int}^{(1)}$ is an SU(3)-scalar. In the following the reduced matrix is absorbed into the coupling constant.

A. SU$_F$(3)-couplings $B \to D + Q$

We consider the tri-quark-baryon, where the baryon is an octet {8}-, the quark a triplet {3}-, and the di-quark a sextet {6}-state. The flavor interaction is

$$\begin{aligned} \mathcal{H}_{int}^{(1)}(x) &= +\lambda_3\left(B^\dagger DQ + h.c.\right) \\ &= +g_3 \left(\begin{array}{ccc} \mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu \end{array}\right) \Psi_\nu^{(\mu)\dagger}(x)\, \chi_{\nu_1}^{(\mu_2)}(x)\Phi_{\nu_2}^{(\mu_1)}(x) + h.c. \end{aligned} \tag{3.5}$$

where for the baryons $\{\mu\} = \{8\}$, the di-quarks $\{\mu_2\} = \{6\}$, and quarks $\{\mu_1\} = \{3\}$. The vertex is

$$\begin{aligned} \Gamma^{\mu_Q\ \mu_D\ \mu_B}_{\nu_Q\ \nu_D\ \nu_B} &= \left(B_\nu^{(\mu)}|\mathcal{H}_{int}^{(1)}|\chi_{\nu_D}^{\mu_D}\phi_{\nu_Q}^{\mu_Q}\right) = \left(\begin{array}{ccc} \mu_Q & \mu_D & \mu_B \\ \nu_Q & \nu_D & \nu_B \end{array}\right) \\ &= g_3\, C\left(\begin{array}{ccc} I_D & I_Q & I_B \\ I_{zD} & I_{zQ} & I_{zB} \end{array}\right) \left(\begin{array}{cc|c} \mu_Q & \mu_D & \mu \\ I_QY_Q & I_DY_D & I_BY_B \end{array}\right) \end{aligned} \tag{3.6}$$

TABLE I: SU(3) Clebsch-Gordan coefficients and Isoscalar (Iscl) factors for $\{8\} \otimes \{3\}$.

Channel:	I,Y	$\nu_Q(I,Y)$	$\nu_B(I,Y)$	C(isospin)	Iscl {15}	Iscl $\{\bar{6}\}$	Iscl {3}	$\{\mu\}$
P +u:	1, $\frac{4}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, 1	1	1	—	—	**15**
P +d:	1, $\frac{4}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, 1	$+\sqrt{\frac{1}{2}}$	1	—	—	**15**
N +u:	1, $\frac{4}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, 1	$\sqrt{\frac{1}{2}}$	1	—	—	**15**
N +u:	0, $\frac{4}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, 1	$+\sqrt{\frac{1}{2}}$	—	-1	—	$\bar{\mathbf{6}}$
P +d:	0, $\frac{4}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, 1	$-\sqrt{\frac{1}{2}}$	—	-1	—	$\bar{\mathbf{6}}$
P +s:	$\frac{1}{2}$, $\frac{1}{3}$	0, $-\frac{2}{3}$	$\frac{1}{2}$, 1	1	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{8}}$	**15**, $\bar{\mathbf{6}}$, **3**
Λ+u:	$\frac{1}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	0 0	1	$\sqrt{\frac{9}{16}}$	$-\sqrt{\frac{3}{8}}$	$-\sqrt{\frac{1}{16}}$	**15**, $\bar{\mathbf{6}}$, **3**
Λ+s:	0, $-\frac{2}{3}$	0, $-\frac{2}{3}$	0 0	1	$\sqrt{\frac{3}{4}}$	—	$\sqrt{\frac{1}{4}}$	**15**, **3**
Σ^++u:	$\frac{3}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	1 0	1	1	—	—	**15**
Σ^++s:	1, $-\frac{2}{3}$	0, $-\frac{2}{3}$	1 0	1	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	—	**15**, $\bar{\mathbf{6}}$
Ξ^0+u:	1, $-\frac{2}{3}$	$\frac{1}{2}$, $\frac{1}{3}$	$\frac{1}{2}$, -1	1	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$	—	**15**, $\bar{\mathbf{6}}$
Ξ^0+s:	$\frac{1}{2}$, $-\frac{5}{3}$	0, $-\frac{2}{3}$	$\frac{1}{2}$, -1	1	1	—	—	**15**

and for the di-quark-exchange matrix elements

$$\left(\mu'_Q,\nu'_Q;\mu'_B,\nu'_B|\mathcal{H}^{(2)}_{int}|\mu_B,\nu_B;\mu_Q,\nu_Q\right) = g_3^2 \begin{pmatrix} \mu'_Q & \mu_D & \mu'_B \\ \nu'_Q & \nu_D & \nu'_B \end{pmatrix} \begin{pmatrix} \mu_Q & \mu_D & \mu_B \\ \nu_Q & \nu_D & \nu_B \end{pmatrix}, \tag{3.7a}$$

$$\begin{aligned}\langle Q_{\nu_{Q,f}} B_{\nu_{B,f}}|M|B_{\nu_{B,i}} Q_{\nu_{Q,i}}\rangle &= \begin{pmatrix} 3 & 8 & \mu_f \\ \nu_{Q,f} & \nu_{\nu_{B,f}} & \nu_f \end{pmatrix} \begin{pmatrix} 8 & 3 & \mu_i \\ \nu_{B,i} & \nu_{Q,i} & \nu_i \end{pmatrix} \cdot \\ &\times \left(\mu'_Q,\nu'_Q;\mu'_B,\nu'_B|\mathcal{H}^{(2)}_{int}|\mu_B,\nu_B;\mu_Q,\nu_Q\right)\end{aligned} \tag{3.7b}$$

where $\mu_i = \mu_f = \{8\},\{10\}$. The CGC for the initial and final state have the form

$$\begin{pmatrix} 3 & 8 & \mu \\ \nu_Q & \nu_B & \nu \end{pmatrix} = C \begin{pmatrix} I_Q & I_B & I \\ I_{zQ} & I_{zB} & I_z \end{pmatrix} \left(\begin{array}{cc|c} 3 & 8 & \mu \\ I_Q\, Y_Q & I_B Y_B & IY \end{array} \right)$$

with $\mu = \{15\},\{\bar{6}\},\{3\}$. In Table I the $SU_I(2)$ Clebsch-Gordan and Isoscalar coefficients are listed for a set of B+Q channels. The expression of the interaction matrix elements in the SU(3) reduced matrix elements is obtained as follows [7]

$$\begin{aligned}\left(\mu'_Q,\nu'_Q;\mu'_B,\nu'_B|\mathcal{H}^{(2)}_{int}|\mu_B,\nu_B;\mu_Q,\nu_Q\right) &= \sum_{\mu_i,\mu_f} \begin{pmatrix} \{3\} & \{8\} & \{\mu_f\} \\ \nu'_Q & \nu'_B & \nu_f \end{pmatrix} \begin{pmatrix} \{8\} & \{3\} & \{\mu_i\} \\ \nu_B & \nu_Q & \nu_f \end{pmatrix} \cdot \\ &\times \left(\{\mu\}_f,\nu_f|\mathcal{H}^{(2)}_{int}|\{\mu\}_i,\nu_i\right)\delta_{\mu_i,\mu_f}\delta_{\nu_i,\nu_f} \\ &= \sum_{\mu,\nu} \begin{pmatrix} \{3\} & \{8\} & \{\mu\} \\ \nu'_Q & \nu'_B & \nu \end{pmatrix} \begin{pmatrix} \{8\} & \{3\} & \{\mu\} \\ \nu_B & \nu_Q & \nu \end{pmatrix} \langle\mu,\nu||\mathcal{H}^{(2)}_{int}||\mu,\nu\rangle\end{aligned} \tag{3.8}$$

To study the anti-symmetrization between the quarks and the baryons we use the tensor-algebra, see [9], for the SU(3)-irreps. For the fundamental representation {3} the basic vectors are $\xi_j = |3,j\rangle$. For the {8} states $\xi_j^k = |8,jk\rangle$ $(j \neq k), \xi_j^j = (\sqrt{2}/3)\left[|8,\rho^{ji}\rangle + |8,\rho^{jk}\rangle\right]$. For the {6} states $\xi_{jk} = |6,jk\rangle$, for the {10} states $\xi_{jkl} = |10,jkl\rangle$, and for the {15} states $\xi^l_{jk} = |15,jkl\rangle$. The QB-states the flavor symmetric and antisymmetric states are $|\psi_{QB}\rangle_\pm = (\xi_j \otimes \xi^k_l \pm \xi^k_l \otimes \xi_j)/\sqrt{2}$. The flavor-spin-orbital group is $SU_f(3) \otimes SU_\sigma(2) \otimes O(3)$, and the anti-symmetry requires $P_f P_\sigma P_x = -1$ in the subspace of allowed QB-states.
The octet-baryons are in the SU(6) {56}-irrep and have the form, see *e.g.* ref. [10],

$$|\mathbf{8},\mathbf{2}\rangle = \frac{1}{\sqrt{2}}\left(\phi_{M,S}\chi_{M,S} + \phi_{M,A}\chi_{M,A}\right) \tag{3.9}$$

We restrict ourselves to the flavor-spin symmetric states and henceforth we wave functions $\phi_{M,S}$ in this calculation. The di-quark wave function is a direct product $\Psi(D) = \Psi(space-spin)\otimes\Psi(flavor)\otimes\Psi(color)$. The flavor di-quark flavor wave functions for the SU(3)-irrep {6} are

$$\chi(uu) = uu,\ \chi(ud) = \frac{1}{\sqrt{2}}\left(ud+du\right),\ \chi(dd) = dd,$$
$$\chi(us) = \frac{1}{\sqrt{2}}\left(us+su\right),\ \chi(ds) = \frac{1}{\sqrt{2}}\left(ds+sd\right),\ \chi(ss) = ss. \tag{3.10}$$

The SU(3) content of the B + Q channels, with $\nu_B = \{Y_B, I_B, I_{Bz}\}$ and $\nu_Q = \{Y_Q, I_Q, I_{Qz}\}$, is

$$\left|B_{\nu_B}, Q_{\nu_Q}; I, M\right\rangle \ = \sum_{\boldsymbol{\mu}=\mathbf{15},\bar{\mathbf{6}},\mathbf{3}} \begin{pmatrix} \mu_B & \mu_Q & \mu \\ \nu_B & \nu_Q & \nu \end{pmatrix} \left|\boldsymbol{\mu}; Y, I\right\rangle = C\begin{pmatrix} I_B & I_Q & I \\ m_B & m_Q & M \end{pmatrix}.$$
$$\times \sum_{\boldsymbol{\mu}=\mathbf{15},\bar{\mathbf{6}},\mathbf{3}} \left(\begin{array}{cc|c} \{8\} & \{3\} & \{\mu\} \\ I_B Y_B & I_Q Y_Q & I_\mu Y_\mu \end{array}\right) \left|\boldsymbol{\mu}; Y, I\right\rangle. \tag{3.11}$$

Application to a set of channels gives, apart from the $SU_I(2)$ Clebsch-Gordan coefficient in (3.11),

$$P+u: Y=\frac{4}{3}, I=1\ :\ \left|\mathbf{15};\frac{4}{3},1\right\rangle,\ N+u: Y=\frac{4}{3}, I=0\ :\ \sqrt{\frac{1}{3}}\left|\bar{\mathbf{6}};\frac{4}{3},0\right\rangle, \tag{3.12a}$$

$$P+s: Y=\frac{1}{3}, I=\frac{1}{2}\ :\ \sqrt{\frac{3}{8}}\left|\mathbf{15};\frac{1}{3},\frac{1}{2}\right\rangle+\sqrt{\frac{1}{4}}\left|\bar{\mathbf{6}};\frac{1}{3},\frac{1}{2}\right\rangle+\sqrt{\frac{3}{8}}\left|\mathbf{3};\frac{1}{3},\frac{1}{2}\right\rangle,$$
$$\Lambda+u: Y=\frac{1}{3}, I=\frac{1}{2}\ :\ \sqrt{\frac{9}{16}}\left|\mathbf{15};\frac{1}{3},\frac{1}{2}\right\rangle-\sqrt{\frac{3}{8}}\left|\bar{\mathbf{6}};\frac{1}{3},\frac{1}{2}\right\rangle-\sqrt{\frac{1}{16}}\left|\mathbf{3};\frac{1}{3},\frac{1}{2}\right\rangle\Big], \tag{3.12b}$$
$$\Lambda+s: Y=-\frac{2}{3}, I=0\ :\ \sqrt{\frac{3}{4}}\left|\mathbf{15};-\frac{2}{3},0\right\rangle+\sqrt{\frac{1}{4}}\left|\mathbf{3};-\frac{2}{3},0\right\rangle\Big],$$

$$\Sigma^{+}+u: Y=\frac{1}{3}, I=\frac{3}{2}\ :\ \left|\mathbf{15};\frac{1}{3},\frac{3}{2}\right\rangle,\ \Sigma^{0}+u: Y=\frac{1}{3}, I=\frac{3}{2}\ :+\sqrt{\frac{2}{3}}\left|\mathbf{15};\frac{1}{3},\frac{1}{2}\right\rangle, \tag{3.12c}$$
$$\Sigma^{0}+u: Y=\frac{1}{3}, I=\frac{1}{2}\ :\ -\sqrt{\frac{1}{3}}\Big[-\sqrt{\frac{1}{16}}\left|\mathbf{15};\frac{1}{3},\frac{1}{2}\right\rangle-\sqrt{\frac{3}{8}}\left|\bar{\mathbf{6}};\frac{1}{3},\frac{1}{2}\right\rangle+\sqrt{\frac{9}{16}}\left|\mathbf{3};\frac{1}{3},\frac{1}{2}\right\rangle\Big],$$

$$\Xi^{-}+u: Y=-\frac{2}{3}, I=1\ :\ -\sqrt{\frac{1}{2}}\Big[\sqrt{\frac{1}{2}}\left|\mathbf{15};-\frac{2}{3},0\right\rangle-\sqrt{\frac{1}{2}}\left|\bar{\mathbf{6}};-\frac{2}{3},0\right\rangle\Big], \tag{3.12d}$$
$$\Xi^{-}+u: Y=-\frac{2}{3}, I=0\ :\ -\sqrt{\frac{1}{2}}\Big[\sqrt{\frac{3}{4}}\left|\mathbf{15};-\frac{2}{3},0\right\rangle+\sqrt{\frac{1}{4}}\left|\bar{\mathbf{6}};-\frac{2}{3},0\right\rangle\Big].$$

Since the interactions respect SU(3)-symmetry it follows that $\mu_f = \mu_i, Y_f = Y_i, I_f = I_i$. For example $\langle\Sigma^0, u|\Lambda, u\rangle = 0$. The symmetry w.r.t. the interchange of the B and Q reads [7]

$$C\begin{pmatrix} I_Q & I_B & I \\ m_Q & m_B & M \end{pmatrix} = (-)^{I_B+I_Q-I} C\begin{pmatrix} I_B & I_Q & I \\ m_B & m_Q & M \end{pmatrix}$$
$$\left(\begin{array}{cc|c} \{3\} & \{8\} & \{\mu\} \\ I_Q Y_Q & I_B Y_B & IY \end{array}\right) = \xi_1 (-)^{I_B+I_Q-I} \left(\begin{array}{cc|c} \{8\} & \{3\} & \{\mu\} \\ I_B Y_B & I_Q Y_Q & IY \end{array}\right),$$

where $\xi_1 = +1, -1, -1$ for $\mathbf{15}, \bar{\mathbf{6}}, \mathbf{3}$ respectively, see [8] Table 2. The expression of the matrix elements $\langle Q, B|\mathcal{L}^{(2)}_{int}|B, Q\rangle$ in terms of the the SU(3) reduced matrix elements $\langle\mu||\mathcal{L}^{(2)}_{int}||\mu\rangle$ in (3.8) can be read off from (3.12).

B. Di-quark-exchange SU$_I$(2) Couplings and Matrix elements.

Introducing the isospin doublets,

$$n = \begin{pmatrix} u \\ d \end{pmatrix},\ K_6 = \sqrt{\frac{1}{2}}\begin{pmatrix} us+su \\ ds+sd \end{pmatrix}, \tag{3.13}$$

the isotriplet $\boldsymbol{\pi}_6 = \big(uu, (ud+du)/\sqrt{2}, dd\big)$, and isosinglets s and $\xi_6 = (ss)$.
The baryon-octet deconfinement interaction reads

$$\mathcal{H}^{(1)}_{int} = g_3 \left(\begin{array}{ccc} \mu_6 & \mu_3 & \mu_8 \\ \nu_6 & \nu_3 & \nu_8 \end{array}\right) \left[\Psi^{(8)\dagger}_{-\nu_8}\chi^{(6)}_{\nu_6}Q^{(3)}_{\nu_3} + h.c.\right] \tag{3.14}$$

The interaction Lagrangians which is are scalar w.r.t. $\mathrm{SU}_I(2)$ are of the form

$$\mathcal{H}^{(1)}_{int}(I_6,Y_6;I_3,Y_3;I_8,Y_8) = g(I_6,Y_6;I_3,Y_3;I_8,Y_8) \sum_{m_6,m_3,m_8} C\left(\begin{array}{ccc} I_6 & I_3 & I_8 \\ m_6 & m_3 & m_8 \end{array}\right) \cdot$$
$$\times\Psi^{(8)\dagger}(I_8,-m_8,-Y_8)\ \chi^{(6)}(I_6,m_6,Y_6)\ Q^{(3)}(I_q,m_q,Y_q) \tag{3.15}$$

With

$$\left(\begin{array}{ccc} \mu_6 & \mu_3 & \mu_8 \\ \nu_6 & \nu_3 & \nu_8 \end{array}\right) = \left(\begin{array}{cc|c} \mu_6 & \mu_3 & \mu_8 \\ I_6Y_6 & I_3Y_3 & I_8Y_8 \end{array}\right) C\left(\begin{array}{ccc} I_6 & I_3 & I_8 \\ m_6 & m_3 & m_8 \end{array}\right)$$

one gets the relations

$$\begin{aligned}\mathcal{H}^{(1)}_{int} &= g_3 \sum_{I_6Y_6;I_3Y_3;I_8Y_8} \left(\begin{array}{cc|c} \mu_6 & \mu_3 & \mu_8 \\ I_6Y_6 & I_3Y_3 & I_8Y_8 \end{array}\right) \cdot \\ &\quad\times\mathcal{H}_{int}(I_6,Y_6;I_3,Y_3;I_8,Y_8)\end{aligned} \tag{3.16}$$

and

$$g(I_6,Y_6;I_3,Y_3;I_8,Y_8) = g_3 \left(\begin{array}{cc|c} \mu_6 & \mu_3 & \mu_8 \\ I_6Y_6 & I_3Y_3 & I_8Y_8 \end{array}\right) \cdot$$

This gives the following SU(3) relations between the couplings

$$\begin{aligned} g_{Nu\pi} &= +g_3,\ g_{\Sigma s\pi} = -\sqrt{\frac{2}{3}}\ g_{Nu\pi},\ g_{\Sigma nK} = \sqrt{\frac{1}{3}}\ g_{Nu\pi},\ g_{\Lambda nK} = g_{Nu\pi}, \\ g_{\Xi sK} &= -g_{Nu\pi},\ g_{\Xi n\xi} = -\sqrt{\frac{2}{3}}\ g_{Nu\pi}. \end{aligned} \tag{3.17}$$

The $\mathrm{SU}_I(2)$-invariant interaction Lagrangian for the coupling of the octet baryons to the di-quarks is

$$\begin{aligned}\mathcal{H}^{(1)}_{int} &= g_{Nn\pi}\big[(N^\dagger\boldsymbol{\tau}n) + (n^\dagger\boldsymbol{\tau}N)\big]\cdot\boldsymbol{\pi}_6 + g_{\Sigma s\pi}\left[(\boldsymbol{\Sigma}^\dagger\ s) + (s^\dagger\boldsymbol{\Sigma})\right]\cdot\boldsymbol{\pi}_6 \\ &\quad+g_{n\Sigma K}\left[\boldsymbol{\Sigma}^\dagger\cdot(K_6^\dagger\boldsymbol{\tau}n) + (n^\dagger\boldsymbol{\tau}K_6)\cdot\boldsymbol{\Sigma}\right] + g_{n\Lambda K}\left[(n^\dagger K_6)\Lambda + \Lambda^\dagger(K_6^\dagger n)\right] \\ &\quad+g_{\Xi sK}\left[(\Xi^\dagger K_6)s + s^\dagger(K_6^\dagger\Xi)\right] + g_{\Xi n\xi}\left[(\Xi^\dagger n) + (n^\dagger\Xi)\right]\ \xi_6. \end{aligned} \tag{3.18}$$

In general for the di-quark-exchange matrix element of the Hamiltonian

$$\mathcal{H}^{(2)}_{QN,QN} = g_3^2\big(\bar{Q}_f\Gamma_1B_f\big)\cdot\big(\bar{B}_f\Gamma_2Q_i\big), \tag{3.19}$$

with $g_3 = \lambda_3/\mathcal{M}$. The matrix element, with $I_f = I_i = I$, and the vertices Γ_1, Γ_2 for respectively the upper and lower vertex in Fig. 3 panel (a), is

$$M(I_f,I_i) = \Big(f\big|\mathcal{H}^{(2)}_{int}\big|i\Big) = +g_{B_iQ_fD_6}g_{B_fQ_iD_6}A(I_f,I_i,i)\ \delta_{I_f,I_i}, \tag{3.20a}$$

$$\begin{aligned} A(I_f,I_i;i) &= \big(I\big|\Gamma_1(i)\cdot\Gamma_2(i)\big|I\big) = (-)^i\sqrt{2i+1}\cdot \\ &\quad\times\left[\begin{array}{ccc} I_1 & I_2 & I_i \\ i & i & 0 \\ I_1' & I_2' & I_f \end{array}\right]\langle I_{Q_f}||\Gamma_1||I_{B_i}\rangle\langle I_{B_f}||\Gamma_2||I_{Q_i}\rangle. \end{aligned} \tag{3.20b}$$

TABLE II: Di-quark-exchange Matrix elements $M(I_f, I_i) = (f|\mathcal{H}^{(2)}_{int}|i)$ for $B+Q \to Q+B$. $I = I_i = I_f$, denotes the total and i the exchanged di-quark isospin. $A(I_f, I_i)$ is the isospin factor. The generalized Pauli-principle requires $P_I P_\sigma P_x = -1$

I	Channel	i	$A(I_f,I_i)$	Coupling	$M(I_f,I_i)$	$\{\mu\}$'s	P_I, P_σ, P_x	$\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$
1	$P+u\to u+P$	1	1	$g^2_{Nn\pi}$	$-g^2_{Nn\pi}$	**15**	+, -, +	–3
0	$N+u\to u+N$	1	-3	$g^2_{Nn\pi}$	$+3g^2_{Nn\pi}$	$\bar{\mathbf{6}}$	-, +, +	+1
1/2	$P+s\to u+\Sigma^0$	1	$+3$	$g_{\Sigma s\pi}g_{Nn\pi}$	$+\sqrt{6}g^2_{Nn\pi}$	**15**, $\bar{\mathbf{6}}$, **3**	+/-,-/+, +	$0^{\star)}$
3/2	$\Sigma^++u\to u+\Sigma^+$	1/2	$+3\sqrt{3}$	$g^2_{\Sigma nK}$	$-\sqrt{3}g^2_{Nn\pi}$	**15**	+, -, +	–3
1/2	$\Sigma^0+u\to u+\Sigma^0$	1/2	$+6\sqrt{3}$	$g^2_{\Sigma nK}$	$-2\sqrt{3}g^2_{Nn\pi}$	**15**, $\bar{\mathbf{6}}$, **3**	+, -, +	–3
1/2	$\Lambda+u\to u+\Lambda$	1/2	$+1/2$	$g^2_{\Lambda nK}$	$-\frac{1}{2}g^2_{Nn\pi}$	**15**, $\bar{\mathbf{6}}$, **3**	+, -, +	–3
1/2	$\Lambda+u\to u+\Sigma^0$	1/2	$+\sqrt{\frac{3}{2}}$	$g_{\Lambda nK}g_{\Sigma nK}$	$-\sqrt{\frac{1}{2}}g^2_{Nn\pi}$	**15**, $\bar{\mathbf{6}}$, **3**	+/-,-/+, +	$0^{\star)}$
0	$\Lambda+s\to s+\Lambda$	0	1	$g^2_{\Lambda sK}$	$-g^2_{Nn\pi}$	**15**, **3**	+, -, +	–3
0	$\Lambda+s\to u+\Xi^-$	1/2	$+\sqrt{2}$	$g_{\Lambda nK}g_{\Xi sK}$	$+\sqrt{2}g^2_{Nn\pi}$	**15**, **3**	+/-,-/+, +	$0^{\star)}$

We evaluate the isospin CGC 9j-coefficients using the relation with the 6j-coefficients

$$\begin{bmatrix} I_1 & I_2 & I_i \\ i & i & 0 \\ I_1' & I_2' & I_f \end{bmatrix} = (-)^{I_1+I_2'+I_i+i}\left[\frac{(2I_1'+1)(2I_2'+1)}{\sqrt{(2i+1)}}\right]^{1/2}\left\{\begin{matrix} I_2 & I_1 & I_i \\ I_1' & I_2' & i \end{matrix}\right\}, \tag{3.21}$$

where index i is the isospin of the exchanged di-quark and $I_i = I_f$ denotes the total isospin. The 9j-symbol $[...]$ is defined in [11] which is related to the 9j-coefficient $\{...\}$ in [12]. The 6j-symbols are tabulated Ref. [12] Table 5.

In Appendix A the details of the evaluation of $A(I_f, I_i; i)$ are worked out for the different channels. In Table II the results are summarized and the matrix elements $M(I_f, I_i) = +g_{B_iQ_fD}\ g_{B_fQ_iD}\ A(I_f, I_i)$ are given as a function of $g^2_{Nn\pi}$. [1] The values for $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ are for S-waves ($P_x = +1$), and -3 for $^1S_0$ and +1 for $^3S_1$. In the cases marked by $\star$) the initial and final states have opposite P_I and P_σ, but the same P_x. Furthermore, the $^1S_0 \leftrightarrow ^3S_1$ transition gives $\langle\chi_f|\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2|\chi_i\rangle = 0$. Consequently for the spin-spin interaction all non-diagonal potentials vanish. This is also the case for the tensor and spin-orbit potentials, only the (small) anti-symmetric spin-orbit survives.

IV. SUMMARY AND CONCLUSION

From Table II the non-zero products $M(I_f, I_i)\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle$ are positive, *i.e.* the spin-spin potentials are repulsive.
In deriving further the di-quark-exchange potentials for baryons Λ, Σ, Ξ, we note that all ingredients and steps are the same for baryons and nucleons. Therefore, all di-quark-exchange potentials $B+Q \to Q+B$ lead to repulsive QB-potentials for the baryon-octet members. This because the effective Lagrangian has the same sign for all baryons.

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[1] The isospin $P_\sigma = (-)^{I_1+I_2+I}$ which follows from $C\begin{pmatrix} I_1 & I_2 & I \\ m_1 & m_2 & m \end{pmatrix} = (-)^{I_1+I_2+I} C\begin{pmatrix} I_2 & I_1 & I \\ m_2 & m_1 & m \end{pmatrix}$.

APPENDIX A: MISCELLANEOUS CALCULATIONS

(a): The i=1 exchange between two I=1/2 particles, *e.g.* $P+u\to u+P$, is

$$\begin{aligned}
A(I_f,I_i) &= \big(I_f,m_f\big|\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\big|I_i,m_i\big) = C\begin{pmatrix} I'_1 & I'_2 & I_f \\ m'_1 & m'_2 & m_f\end{pmatrix} C\begin{pmatrix} I_1 & I_2 & I_i \\ m_1 & m_2 & m_i\end{pmatrix}\cdot \\
&\quad \times\sum_m (-)^m \left(I'_2,m'_2\big|\tau_m\big|I_2,m_2\right)\left(I'_1,m'_1\big|\tau_{-m}\big|I_1,m_1\right) \\
&= -3\sqrt{3}C\begin{pmatrix} I'_1 & I'_2 & I_f \\ m'_1 & m'_2 & m_f\end{pmatrix} C\begin{pmatrix} I_1 & I_2 & I_i \\ m_1 & m_2 & m_i\end{pmatrix}\cdot \\
&\quad \times C\begin{pmatrix} I_2 & 1 & I'_2 \\ m_2 & m & m'_2\end{pmatrix} C\begin{pmatrix} 1 & 1 & 0 \\ m & -m & 0\end{pmatrix} C\begin{pmatrix} I_1 & 1 & I'_1 \\ m_1 & -m & m'_1\end{pmatrix} \\
&= -3\sqrt{3}\begin{bmatrix} I_1 & I_2 & I_i \\ 1 & 1 & 0 \\ I'_1 & I'_2 & I_f\end{bmatrix} = -3\sqrt{3}\left[(2I'_1+1)(2I'_2+1)(2I_i+1)\right]^{1/2}\begin{Bmatrix} I_1 & I_2 & I_i \\ 1 & 1 & 0 \\ I'_1 & I'_2 & I_f\end{Bmatrix} \\
&= -3\sqrt{3}(-)^{I_1+I'_2+I_i+i}\ \left[(2I'_1+1)(2I'_2+1)\right]^{1/2}\begin{Bmatrix} I_2 & I_1 & I_i \\ I'_1 & I'_2 & 1\end{Bmatrix}\ \delta_{I_f,I_i}. \qquad \text{(A1)}
\end{aligned}$$

Here, we used the reduced matrix element $(1/2||\tau||1/2)=\sqrt{3}$,
(a1) For i=1 exchange in Neutron-u-quark system the matrix elements $M(I_f,I_i)$

$$A(1,1) = -3\sqrt{3}\begin{bmatrix} \frac12 & \frac12 & 1 \\ 1 & 1 & 0 \\ \frac12 & \frac12 & 1\end{bmatrix} = +1,\ A(0,0) = -3\sqrt{3}\begin{bmatrix} \frac12 & \frac12 & 0 \\ 1 & 1 & 0 \\ \frac12 & \frac12 & 0\end{bmatrix} = -3. \qquad \text{(A2)}$$

(a2): The i=1 exchange between two I=1/2 particles, *e.g.* $P+S\to U+\Sigma^0$, $I_1=1/2, I_2=0, I'_1=1/2, I'_2=1, i=1$, is

$$\begin{aligned}
A(I_f,I_i) &= \left(I_f,m_f\big|(-)^m\Sigma^\dagger_m\pi_{-m}|I_i,m_i\right) = \\
&= -3C\begin{pmatrix} I'_1 & I'_2 & I_f \\ m'_1 & m'_2 & m_f\end{pmatrix} C\begin{pmatrix} I_1 & I_2 & I_i \\ m_1 & m_2 & m_i\end{pmatrix}\cdot \\
&\quad \times C\begin{pmatrix} I_2 & 1 & I'_2 \\ m_2 & m & m'_2\end{pmatrix} C\begin{pmatrix} 1 & 1 & 0 \\ m & -m & 0\end{pmatrix} C\begin{pmatrix} I_1 & 1 & I'_1 \\ m_1 & -m & m'_1\end{pmatrix} \\
&= -3\begin{bmatrix} I_1 & I_2 & I_i \\ 1 & 1 & 0 \\ I'_1 & I'_2 & I_f\end{bmatrix} = -3\left[(2I'_1+1)(2I'_2+1)(2I_i+1)\right]^{1/2}\begin{Bmatrix} I_1 & I_2 & I_i \\ 1 & 1 & 0 \\ I'_1 & I'_2 & I_f\end{Bmatrix} \\
&= -3(-)^{I_1+I'_2+I_i+i}\ \left[(2I'_1+1)(2I'_2+1)\right]^{1/2}\begin{Bmatrix} I_2 & I_1 & I_i \\ I'_1 & I'_2 & i\end{Bmatrix}\ \delta_{I_f,I_i} \\
A(\frac12,\frac12) &= +3\sqrt{6}\begin{Bmatrix} 0 & \frac12 & \frac12 \\ \frac12 & 1 & 1\end{Bmatrix} = 3. \qquad \text{(A3)}
\end{aligned}$$

(b): The i=1/2 exchange between particles with isospin $I_1=1, I_2=1/2$, *e.g.* $\Sigma^+ + u\to u+\Sigma^+$, The K-exchange interaction Lagrangian reads

$$\mathcal{L}(\Sigma N_q K) = g_{\Sigma QK}\big\{\boldsymbol{\Sigma}^\dagger\cdot(K_c^\dagger\boldsymbol{\tau}N_q) + (N_q^\dagger\boldsymbol{\tau}K)\cdot\boldsymbol{\Sigma}\big\}, \qquad \text{(A4)}$$

with the isospinors $N_q = (u,d), K = (K^+, K^0)$. For $\Sigma^+ + u \to u + \Sigma^+$, $I_1 = I_2' = 1, I_2 = I_1' = 1/2$, we obtain

$$\begin{aligned} A(I_f, I_i) &= \big(I_f, m_f \big| \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \big| I_i, m_i\big) = -3\sqrt{3}\ (-)^{I_1+I_2'+I_i+i}\ \left[(2I_1'+1)(2I_2'+1)\right]^{1/2} \left\{ \begin{array}{ccc} I_2 & I_1 & I_i \\ I_1' & I_2' & 1 \end{array} \right\} \delta_{I_f,I_i} \\ &= +9\sqrt{2} \left\{ \begin{array}{ccc} 1 & \frac{1}{2} & I_i \\ \frac{1}{2} & 1 & 1 \end{array} \right\} \delta_{I_f,I_i}. \end{aligned} \tag{A5}$$

Here, we used again the reduced matrix element $(1/2||\tau||1/2) = \sqrt{3}$, We obtain

$$A(\frac{3}{2},\frac{3}{2}) = +9\sqrt{2} \left\{ \begin{array}{ccc} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & 1 \end{array} \right\} = 3\sqrt{3},\ A(\frac{1}{2},\frac{1}{2}) = -9\sqrt{2} \left\{ \begin{array}{ccc} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 \end{array} \right\} = 6\sqrt{3}. \tag{A6}$$

(c): The i=1/2 exchange between particles with isospin $I_1 = 0, I_2 = 1/2$, *e.g.* $\Lambda + U \to u + \Lambda$. The K-exchange interaction Lagrangian reads

$$\mathcal{L}(\Lambda N_q K) = g_{\Lambda QK} \big\{ \Lambda^\dagger \cdot (K_c^\dagger N_q) + (N_q^\dagger K) \cdot \Lambda \big\}, \tag{A7}$$

(c1): For $\Lambda + U \to U + \Lambda$, $I_1 = I_2' = 0, I_2 = I_1' = 1/2, i = 1/2$, we obtain

$$\begin{aligned} A(I_f, I_i) &= \big(I_f, m_f \big| 1 \big| I_i, m_i\big) = (-)^{I_1+I_2'+I_i+i}\ \left[(2I_1'+1)(2I_2'+1)\right]^{1/2} \left\{ \begin{array}{ccc} I_2 & I_1 & I_i \\ I_1' & I_2' & i \end{array} \right\} \delta_{I_f,I_i} \\ A(\frac{1}{2},\frac{1}{2}) &= -\sqrt{2} \left\{ \begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{array} \right\} = +\frac{1}{2}. \end{aligned} \tag{A8}$$

(c2): For $\Lambda + S \to S + \Lambda$, $I_1 = I_2 = I_1' = I_2' = 0, i = 0$, we obtain

$$A(0,0) = \big(I_f, m_f \big| 1 \big| I_i, m_i\big) = 1. \tag{A9}$$

(c3): For $\Lambda + U \to U + \Sigma^0$, $I_1 = 0, I_2 = 1/2, I_1' = 1/2, I_2' = 1, i = 1/2$, we obtain

$$\begin{aligned} A(I_f, I_i) &= \big(I_f, m_f \big| 1 \big| I_i, m_i\big) = (-)^{I_1+I_2'+I_i+i}\ \left[(2I_1'+1)(2I_2'+1)\right]^{1/2} \left\{ \begin{array}{ccc} I_2 & I_1 & I_i \\ I_1' & I_2' & i \end{array} \right\} \delta_{I_f,I_i} \\ A(\frac{1}{2},\frac{1}{2}) &= +\sqrt{6} \left\{ \begin{array}{ccc} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{array} \right\} = +\sqrt{\frac{3}{2}} \end{aligned} \tag{A10}$$

(c4): For $\Lambda + S \to U + \Xi^-$, $I_1 = 0, I_2 = 0, I_1' = 1/2, I_2' = 1/2, i = 1/2$, we obtain

$$\begin{aligned} A(I_f, I_i) &= \big(I_f, m_f \big| 1 \big| I_i, m_i\big) = (-)^{I_1+I_2'+I_i+i}\ \left[(2I_1'+1)(2I_2'+1)\right]^{1/2} \left\{ \begin{array}{ccc} I_2 & I_1 & I_i \\ I_1' & I_2' & i \end{array} \right\} \delta_{I_f,I_i} \\ A(0,0) &= -2 \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right\} = -2 \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{array} \right\} = +\sqrt{2}. \end{aligned} \tag{A11}$$

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The relation between the 9j-symbol in CGC's and 3j-symbols is

$$\left[\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{array}\right] = [(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)]^{1/2} \left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{array}\right\}$$