Finite Cut-Off Effects and Renormalization in the NN interaction

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Relevant papers

- Phys. Lett. B **580**, 149 (2004) [arXiv:nucl-th/0306069].
- Phys. Rev. C **70**, 044006 (2004) [arXiv:nucl-th/0405057].
- arXiv:nucl-th/0407113. (Low energy np parameters with $j \leq 5$)
- arXiv:nucl-th/0410020. (OPE+TPE Singlet)
- arXiv:nucl-th/0504067. (OPE deuteron)
- arXiv:nucl-th/0506047 (TPE deuteron)

OUTLINE

1. Introduction and motivation
2. Scattering of Neutral Atoms
3. Scattering of Nucleons
4. OPE renormalization of the Deuteron
5. Chiral TPE renormalization of the Deuteron
6. Conclusions

Introduction

What do we know that we do not know ?

The Effective Field Theory (EFT) Dictum

Low energy wave phenomena do not depend on short distance details

Is this copyright of EFT ?

- Basis of Renormalization (common sense). The wavelength λ is the resolution scale, we only see $\Delta r > \lambda$.
- Model Independent Results :

  We incorporate the known long range forces by a *local* potential and parameterize the unknown short distance forces by low energy coefficients.

  This is particularly useful in Nuclear Physics where the interaction at short distances is poorly known.
- Short distance ambiguities should be under control.

  Elimination of short distance/large momentum cut-offs. All calculations should give the same result results, regardless of the regularization method.

Relevant questions

- Can we always assume any parameterization of the short distance physics ? (Restrictions when going to zero range)
- Is this always true for a singular potential ? (Not necessarily)
- Is it compatible with naive dimensional power counting ? (Perhaps not)

Motivation

The chiral NN potentials for heavy baryons in coordinate space (Brockman, Kaiser, Weise , 1997)

$$U(r) = \frac{M_N m_\pi^3}{f_\pi^2} W_{\rm LO}(m_\pi r, g_A) + \frac{M_N m_\pi^5}{f_\pi^4} W_{\rm NLO}(m_\pi r, g_A) + \frac{m_\pi^6}{f_\pi^4} W_{\rm NNLO}(m_\pi r, g_A, c_1, c_3, c_4) + \dots$$

At long distances

$$W_{\rm LO}(mr) \to \frac{e^{-mr}}{mr}$$

$$W_{\rm NLO}(mr) \to \frac{e^{-2mr}}{(mr)^n}$$

$$W_{\rm NNLO}(mr) \to \frac{e^{-2mr}}{(mr)^n}$$

At short distances

$$W_{\rm LO}(mr) \to \frac{1}{(mr)^3}$$

$$W_{\rm NLO}(mr) \to \frac{1}{(mr)^5}$$

$$W_{\rm NNLO}(mr) \to \frac{1}{(mr)^6}$$

We have singular potentials !! $\to$ Special treatment

Scattering of Neutral Atoms

Atom-Atom scattering at low energies

(Ultracold gases, Bose-Eintein situation)

The typical de-Brogie wave length much larger $\lambda \sim 1000 a.u. \gg 1 a.u.$ than typical atom size $\sim 1. a.u.$

The interaction at long distances behaves as a Van der Waals force

$$V(r) = -\frac{C_6}{r^6} - \frac{C_8}{r^8} - \frac{C_{10}}{r^{10}} - \dots$$

The Van der Waals coefficients are calculated from first principles atomic structure calculations.

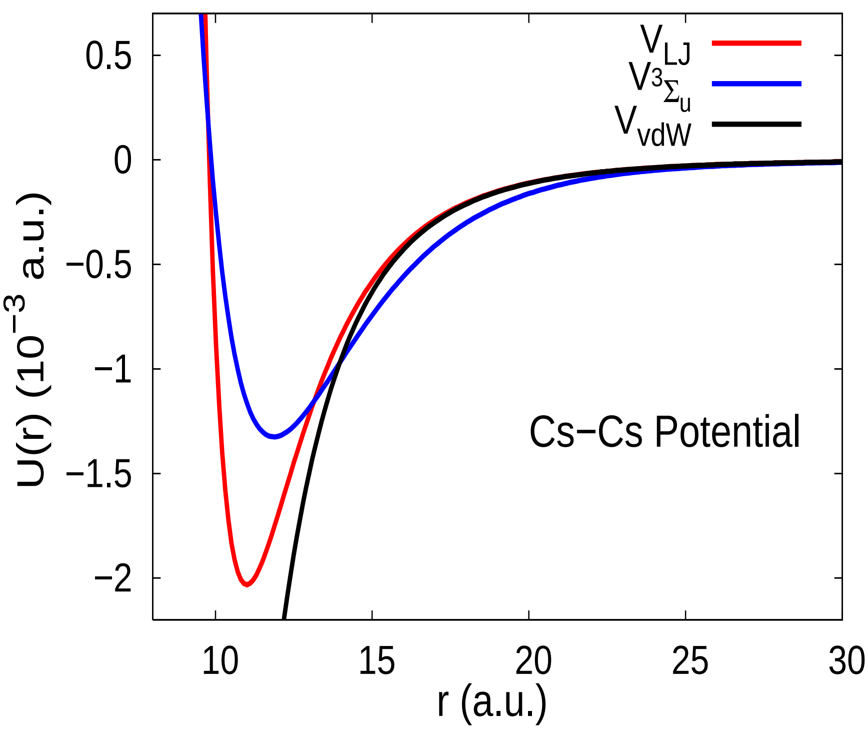
$V(r)$ attractive (Casimir effect : standing waves have lower zero point energy than free waves)

The reduced potential

$$U(r) = 2\mu V(r) = -\frac{R_6^4}{r^6} - \frac{R_8^6}{r^8} - \frac{R_{10}^8}{r^{10}} - \dots$$

Quantum mechanically the system is unstable for $n \geq 2$

$$E_{\rm KIN} \sim \frac{\hbar^2}{2M\Delta r^2} - \frac{C_n}{r^n}$$



Some questions

- Do we expect physically that for $\lambda \gg R_6$ the singularity becomes important for the phase shift ? NO
- Does it make sense to extrapolate the long distance potential down to the origin knowing it is not the "true" potencial ?

  If not, what else ? $\rightarrow$ Cut-off , modelling

  If yes, how ? $\rightarrow$ Renormalization , model independence
- Can we make a model independent prediction so that we need not specify the method of calculation but only the some relevant physical information ?
- Can we truncace the expansion in the potential and extend the range of validity ?
- Can we see the Van der Waals force ?

The short distance theory

For $r \gg R_6$ we neglect the potential.

$$-u_k'' = k^2 u_k$$

The solution is

$$u_k(r) = \frac{\sin(kr + \delta_0)}{\sin \delta_0}$$

We extrapolate to short distances

$$\frac{u_k'(0)}{u_k(0)} = k \cot \delta_0(k)$$

Orthogonality of solutions $0 \le r \le \infty$

$$\delta(k - k') = \int_0^\infty u_k(r) u_{k'}(r) dr$$

implies

$$0 = u_k' u_k - u_k u_k' \Big|_0$$

Thus

$$\frac{u_{k'}'(0)}{u_{k'}(0)} = k \cot \delta_0(k') = \frac{u_k'(0)}{u_k(0)} = k \cot \delta_0(k)$$

Energy independent boundary condition. Taking $k = 0$ and $\delta_0(k) \to -\alpha_0 k$

$$\frac{u_k'(0)}{u_k(0)} = k \cot \delta_0(k) = -\frac{1}{\alpha_0}$$

This is the effective range expansion

$$k \cot \delta_0 = -\frac{1}{\alpha_0} + \frac{1}{2} r_0 k^2 + v_2 k^4 + \dots$$

with

$$r_0 = v_2 = \cdots = 0$$

For $kR_6 \ll 1$ we do not expect to see the long range physics and we can represent it by a boundary condition at the origin.

Long distance perturbation theory

S-wave scattering

$$-u'' + U(r)u = k^2 u \tag{1}$$

The asymptotic solution reads

$$u(r) \rightarrow \sin(kr + \delta(k)) \tag{2}$$

At low energies we have effective range theory

$$k \cot \delta(k) = -\frac{1}{\alpha} + \frac{1}{2} r_0 k^2 + \dots \tag{3}$$

At long distances we can apply perturbation theory. At first order

$$k \cot \delta(k) = k \cot \delta_S + \int_{R_S}^{\infty} dr U(r) \left[\cot \delta_S \sin(kr) + \cos(kr)\right]^2 + \dots$$

δ_S is the phase shift due to the potential in the region $r < R_S \rightarrow 0$ where the interaction is unknown.

Effective range expansion for the short distance phase shift

$$\left.\frac{u_k'(R_S)}{u_k(R_S)}\right|_{R_S \to 0} = k \cot \delta_S(k) = -\frac{1}{\alpha_S} \tag{4}$$

Subtracting at first order, we can remove the regulator

$$k \cot \delta(k) = -\frac{1}{\alpha} + \int_0^{\infty} dr U(r) \left(\left[-\frac{1}{\alpha} \sin(kr) + \cos(kr) \right]^2 - \left[1 - \frac{r}{\alpha} \right]^2 \right) + \dots \tag{5}$$

Then the effective range is predicted

$$r_0 = 4 \int_0^{\infty} dr r^2 U(r) \left(1 - \frac{r}{\alpha_0} \right)^2 \tag{6}$$

Power suppression at the origin when α_0 is fixed.

- r^2 power suppression for α_0 large (short distance dominance)
- r^4 power suppression for α_0 small (long distance dominance)

Short distance contribution to the effective range violates orthogonality (Not all counterterms are compatible, $Q-$counting).

Divergent for a singular potential $\rightarrow$ No perturbative treatment !!

Renormalization of singular potentials. One channel case (Case, 1950)
Reduced Schrödinger equation for angular momentum l is

$$-u'' + U(r)u + \frac{l(l+1)}{r^2} = k^2 u \tag{7}$$

The asymptotic solution reads

$$u(r) = \sin(kr + \delta_l(k)) \tag{8}$$

Power law behaviour at the origin

$$U(r) = \pm(R/r)^n/R^2$$

Local de Broglie wavelength is given by

$$k(r) = \sqrt{|U(r)|}$$

Applicability condition for the WKB approximation $k'(r) \ll 1$
$r \ll R(n/2)^{2/(2+n)}$ one has a semiclassical wave function

$$u_A(r) \to C_A \left(\frac{r}{R}\right)^{n/4} \sin\left[\frac{2}{n-2}\left(\frac{R}{r}\right)^{\frac{n}{2}-1} + \varphi\right] \tag{9}$$

$$\text{for} \qquad U_A \to -\frac{1}{R^2}\left(\frac{R}{r}\right)^n \tag{10}$$

$$u_R(r) \to C_R \left(\frac{r}{R}\right)^{n/4} \exp\left[-\frac{2}{n-2}\left(\frac{R}{r}\right)^{\frac{n}{2}-1}\right] \tag{11}$$

$$\text{for} \qquad U_R \to +\frac{1}{R^2}\left(\frac{R}{r}\right)^n \tag{12}$$

C_A and C_R are normalization constants φ an arbitrary short distance phase.

Orthogonality of states with different energy (positive or negative)

$$\begin{aligned} 0 &= u_k' u_p - u_p u_k' \Big|_0 \\ &= \frac{1}{R}\sin\left(\varphi(k) - \varphi(p)\right) \end{aligned} \tag{13}$$

The short distance phase φ becomes energy independent. So we can use zero energy. $\delta_0 \to -\alpha_0 k$, $\sigma = 4\pi\alpha_0^2$

Scattering states

$$u_0(r) \to 1 - \frac{r}{\alpha_0} \qquad (r \to \infty) \tag{14}$$

$$u_0(r) \to C \left(\frac{r}{R}\right)^{n/4} \sin\left[\frac{2}{n-2}\left(\frac{R}{r}\right)^{\frac{n}{2}-1} + \varphi_0\right] \quad (r \to 0) \tag{15}$$

Steps to follow

- For $k = 0$ we fix α_0
- We integrate in and obtain the short distance phase φ_0
- For $k \neq 0$ we use $\varphi_k = \varphi_0$ (orthogonality constraint)
- We integrate out and obtain δ_0

Then we predict δ_0 from the potential and the scattering length α_0 (dimensional transmutation).

Bound states

$$u_\gamma(r) \to e^{-\gamma r} \qquad (r \to \infty) \tag{16}$$

$$u_\gamma(r) \to C \left(\frac{r}{R}\right)^{n/4} \sin\left[\frac{2}{n-2}\left(\frac{R}{r}\right)^{\frac{n}{2}-1} + \varphi_0\right] (r \to 0) \tag{17}$$

- We fix the bound state energy $E = -\gamma^2/2\mu$
- We integrate in and obtain the phase $\varphi_\gamma = \varphi_0$
- For $k \neq i\gamma$ we use $\varphi_k = \varphi_0$ (orthogonality constraint)
- We integrate out and obtain δ_0

Then we predict δ_0 and the scattering length α_0 from γ

Repulsive case

There is no dimensional transmutation since the regularity and orthogonality condition are fulfilled.

Atom-Atom Scattering (neutral atoms)

Analytical effective range for Van der Waals force $U = -R^4/r^6)$, Gao 1998; Gribakin-Flambaum 1999

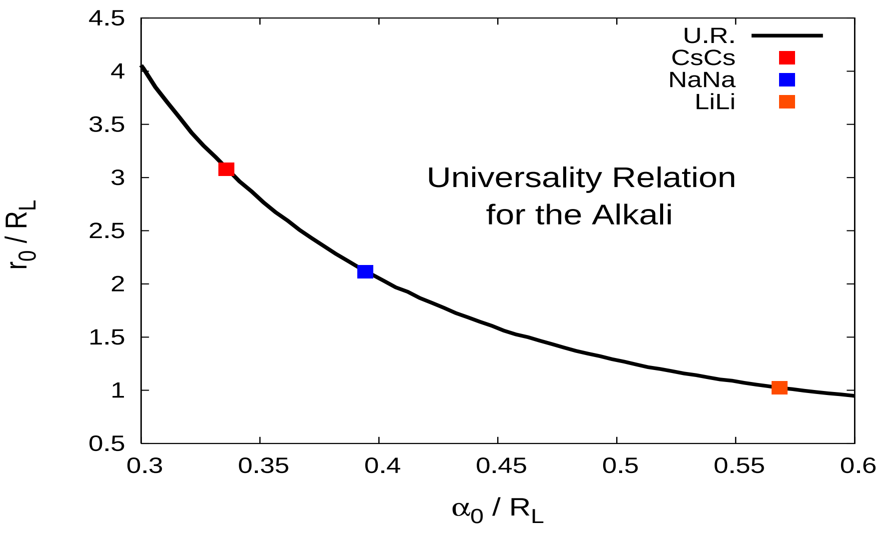
$$
\begin{aligned}
r_0 &= \frac{-4\,R^2}{3\,\alpha} + \frac{4\,R^3\,\Gamma(\frac{3}{4})^2}{3\,\alpha_0^2\,\pi} + \frac{16\,R\,\Gamma(\frac{5}{4})^2}{3\,\pi} \\
&= 1.39473\,R - \frac{1.33333\,R^2}{\alpha_0} + \frac{0.637318\,R^3}{\alpha_0^2} \qquad (18)
\end{aligned}
$$

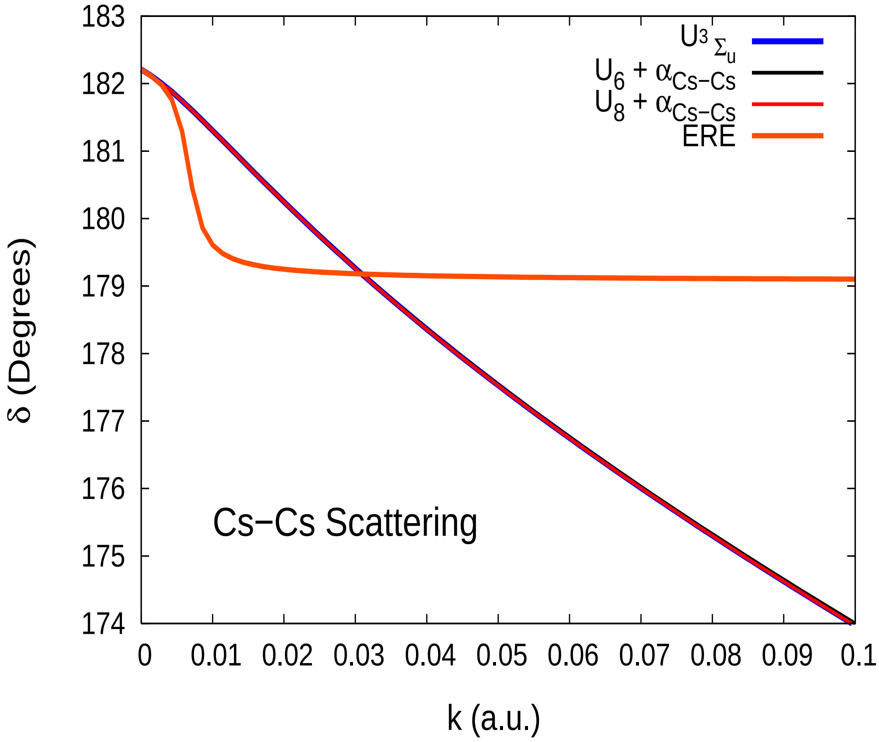
For alkali atoms $R \sim 200a.u.$, $\alpha = 50 - 100$.
Long distance dominance. Only C_6 is needed.

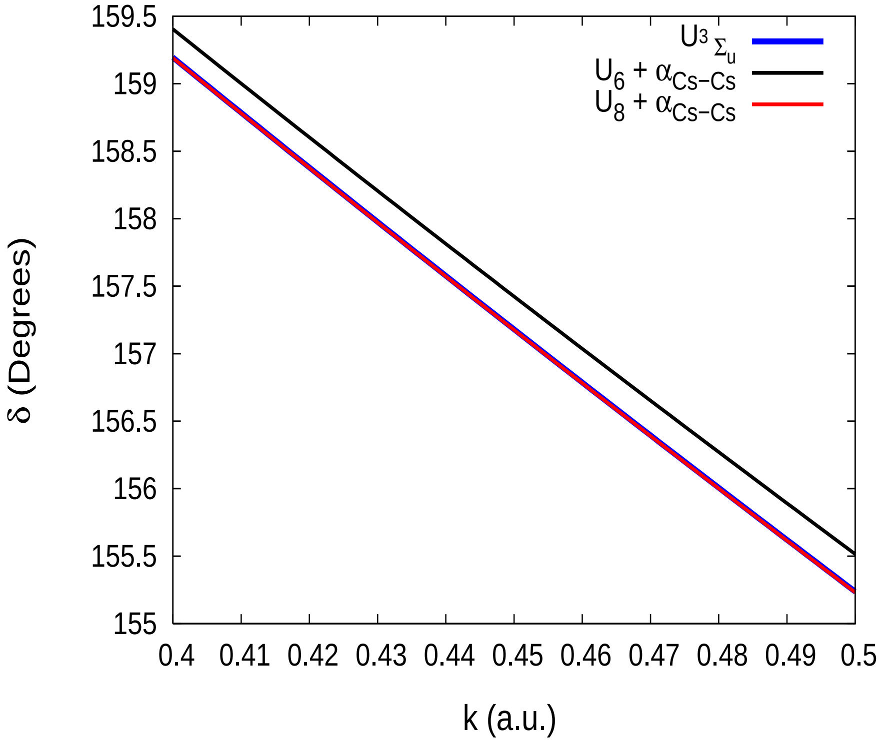
For (Hidrogen, Helium 3 and 4) $R \sim 10a.u.$, $\alpha = 50 - 100$
Short distance dominance. Other coefficients C_8 are needed.
But a possible problem ...

- If one C_n was repulsive the whole effective theory does not make sense.
- In Hidrogen C_{32} all are attractive $\rightarrow$ we can always take α_0 as an input parameter
- Theorem (Lieb-Thirring, 1986) For Coulomb systems

$$
U(r) > -\frac{C_6}{r^6} + \text{const}
$$







Cs-Cs scattering

$$\alpha_0 = 68.13 \qquad R_6 = 203.62 \qquad R_8 = 80.22 \qquad R_{10} = 50.33$$

Scale separation

$$(R_8/R_6)^2 = .15 \ll 1 \quad \rightarrow \quad \delta r_0/r_0 \ll 1$$

Scattering of Nucleons (Singlet s-wave)

$^1S_0$ nucleon-nucleon scattering.

At short distances the NN chiral potential behaves

$$U_{^1S_0}\Big|_{\rm LO} = -\frac{mMg_A^2 e^{-mr}}{16\pi f_\pi^2 r} \to -\frac{mMg_A^2}{16\pi f_\pi^2 r} = -\frac{1}{R_1 r}\,,$$

$$U_{^1S_0}\Big|_{\rm NLO} \to \frac{M(1+10g_A^2-59g_A^4)}{256\pi^3 f_\pi^4 r^5} = -\frac{R_5^3}{r^5}\,,$$

$$U_{^1S_0}\Big|_{\rm NNLO} \to \frac{3g_A^2}{128 f_\pi^4 \pi^2 r^6}(-4+15g_A^2+24c_3M-8c_4M) = -\frac{R_6^4}{r^6}$$

Chiral couplings

- πN (Buttikker-Meissner, 2000)

$$c_1=-0.81\pm 0.15 \qquad -4.69\pm 1.34 \qquad c_4 = 3.40\pm 0.04$$

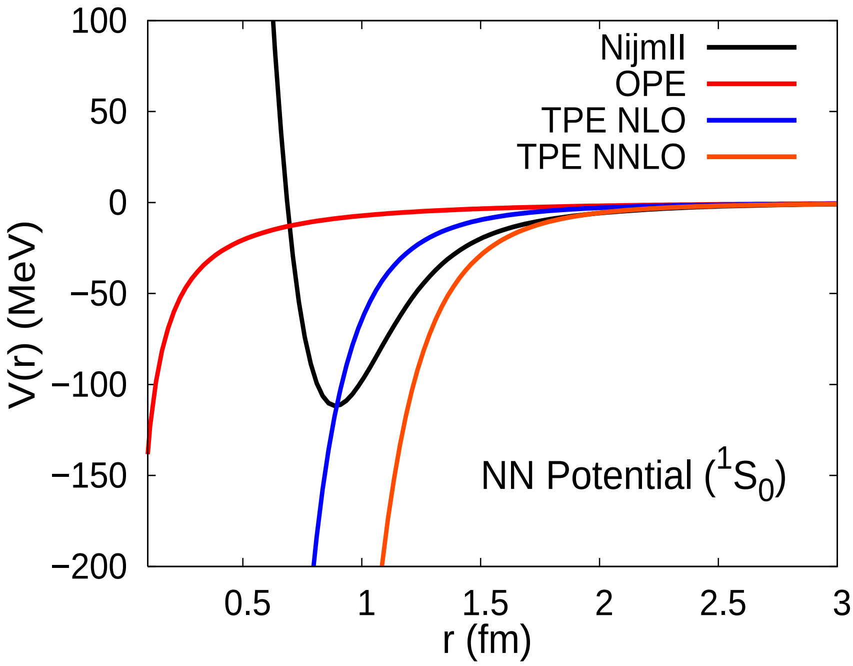
- NN (Rentmeester, Timmermans, Friar, de Swart 1999)

$$c_1=-0.76\pm 0.07 \qquad c_3=-5.08\pm 0.24 \qquad c_4 = 4.70\pm 0.70$$

- NN (Entem,Machleidt, 2003)

$$c_1=-0.81 \qquad c_3=-3.20 \qquad c_4 = 5.40$$

The coefficient c_1 does not appear in the leading singularity. This is general and explains the observed weak dependence on c_1



Universal relations in the single channel case

For the zero energy state we use the normalization

$$u_0(r) \to 1 - \frac{r}{\alpha_0} \tag{19}$$

Then, the effective range is given by

$$r_0 = 2\int_0^\infty dr \left[\left(1 - \frac{r}{\alpha_0}\right)^2 - u_0(r)^2 \right] \tag{20}$$

We can use the superposition principle for boundary conditions

$$u_0(r) = u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r) \tag{21}$$

where $u_{0,c}(r) \to 1$ and $u_{0,s}(r) \to r$

$$r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2} \tag{22}$$

where

$$A = 2\int_0^\infty dr (1 - u_{0,c}^2) \tag{23}$$

$$B = -4\int_0^\infty dr (r - u_{0,c} u_{0,s}) \tag{24}$$

$$C = 2\int_0^\infty dr (r^2 - u_{0,s}^2) \tag{25}$$

Thus the effective range is a quadratic function of the inverse scattering length.

1) Large $\alpha_0 \gg R$ then $r_0 \sim R$ (short distance dominate)

2) Small $\alpha_0 \ll R$ then $r_0 \sim R^3/\alpha_0^2$ (large distance dominate)

Phase shift (explicit separation of potential and scattering length)

$$k \cot \delta_0 = \frac{\alpha_0 \mathcal{A}(k) - \mathcal{B}(k)}{\alpha_0 \mathcal{C}(k) - \mathcal{D}(k)}$$

Functions depending on potential only.

The obvious conditions $\mathcal{A}(0) = \mathcal{D}(0) = 0$ and $\mathcal{B}(0) = \mathcal{C}(0) = 1$ are satisfied.

Expanding the expression for small k one gets that v_k is a polynomial in $1/\alpha_0$ of degree $k+1$.

Numerically we find the following relations in the singlet channel

$$r_0 = 1.39962 - \frac{4.43994}{\alpha_0} + \frac{5.22466}{\alpha_0^2} \qquad \text{(LO)}$$
$$r_0 = 2.6718 - \frac{5.79989}{\alpha_0} + \frac{6.06246}{\alpha_0^2} \qquad \text{(NNLO)} \tag{26}$$

There is no cut-off here.

That means that TPE effects cannot be the full story. Data can be described but are inconsistent with πN constants c_3 and c_4.

Convergence of in the chiral expansion of the potential

$$\alpha_0 = -23.74 \qquad \text{(Input)}$$

- Pionless

$$r_0 = 0 \qquad v_2 = 0 \qquad v_3 = 0 \qquad v_4 = 0$$

- LO

$$r_0 = 1.36 \qquad v_2 = -2.03 \qquad v_3 = 9.29 \qquad v_4 = -50.60$$

- NLO

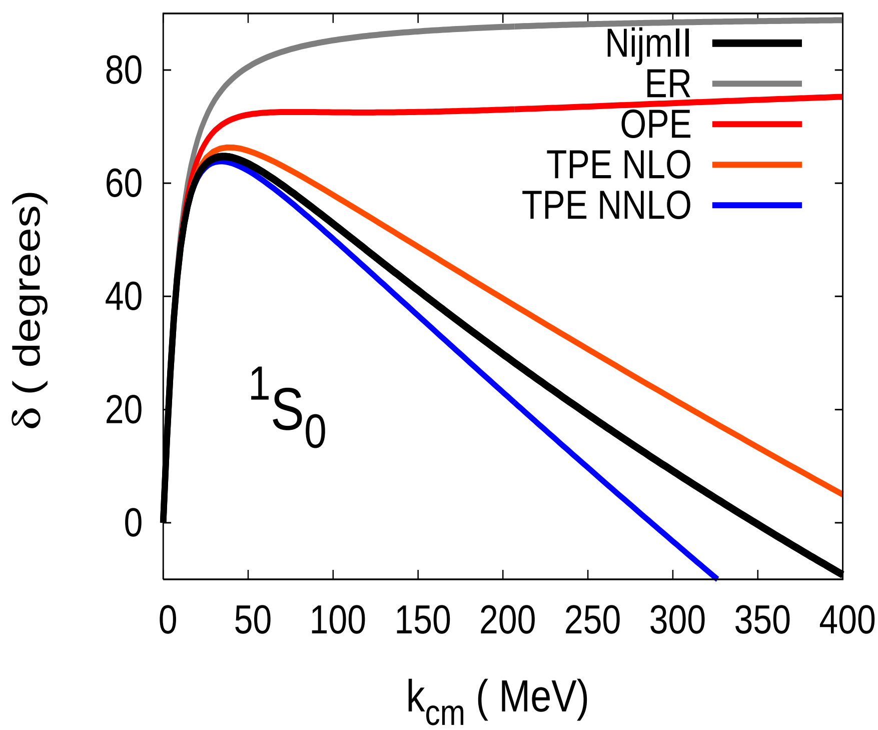
$$r_0 = 2.28 \qquad v_2 = -1.02 \qquad v_3 = 6.09 \qquad v_4 = -35.16$$

- NNLO (Set IV)

$$r_0 = 2.82 \qquad v_2 = -0.36 \qquad v_3 = 4.86 \qquad v_4 = -27.64$$

- Experiment (or NimjII/Reid93)

$$r_0 = 2.77(5) \qquad v_2 = -0.48(1) \qquad v_3 = 3.8(2) \qquad v_4 = -19.5(5)$$



The Wigner causality condition (Feshbach, Lommon 1967)

- Given a local potential which we extrapolate down to the origin, Is the effective range an arbitrary parameter ? NO !!
- Orthogonality imposes that if we switch off the potential completely the short distance distance contribution to the effective range vanishes. (Energy independent boundary condition)
- Causality imposes variational constraints

Long distance chiral potentials are local above some boundary radius.

$$-u''(r) + U(r)u(r) = k^2 u(r) \qquad r > R$$

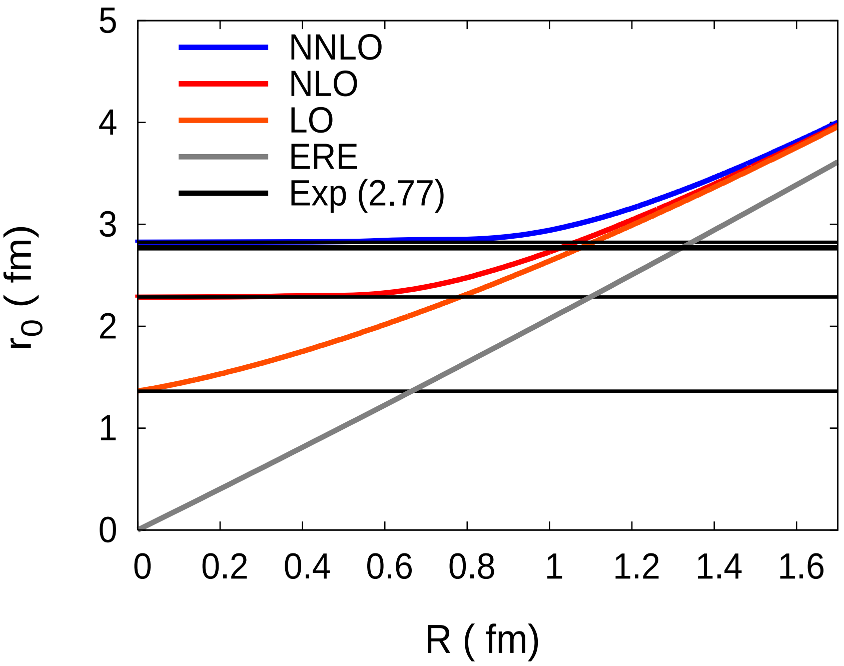
Assume that the physics is non local below a given boundary

$$-u''(r) + \int_0^R dr' U(r,r') u(r') = k^2 u(r) \qquad r < R$$

Wigner bound on the effective range (Phillips, Cohen 1998)

$$r_0(R) \le 2R \left[1 - \frac{R}{\alpha_0(R)} + \frac{R^2}{3\alpha_0(R)} \right]$$

$\alpha_0(R)$ and $r_0(R)$ are parameters when the local potential is switched off from ∞ to R



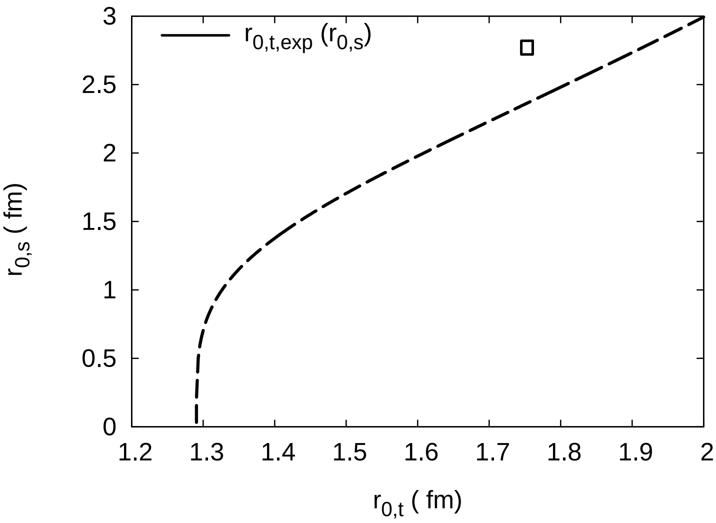
Finite Cut-Off Effects

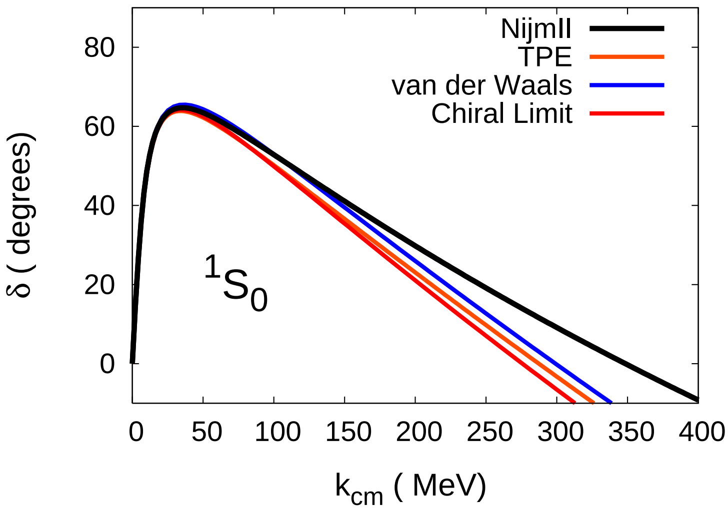
Set	Calculation	$\Lambda = 1/a$	α_0(fm)	r_0(fm)	v_2(fm^3)
Set I	NNLO Epelbaum 1999	0.6-1 GeV	-23.72	2.68	-0.61
Set I	NNLO Entem 2001	0.5 GeV	-23.75	2.70	-
Set I	This work	∞	Input	2.92	-0.30
Set II	NNLO Rentmeester 1999	1/1.4 fm	-	-	-
Set II	This work	∞	Input	2.97	-0.23
Set III	NNLO Epelbaum 2003	0.65 GeV	-23.4	2.67	-0.50
Set III	N^3LO Epelbaum 2004	0.7 GeV	-23.6	2.66	-0.50
Set III	This work	∞	Input	2.83	-0.43
Set IV	N^3LO Entem 2003	0.5 GeV	-23.73	2.73	-
Set IV	This work	∞	Input	2.87	-0.38
Nijm II		-	-23.73	2.67	-0.48
Reid 93		-	-23.74	2.75	-0.49
Exp.	–	–	-23.74(2)	2.77(5)	-

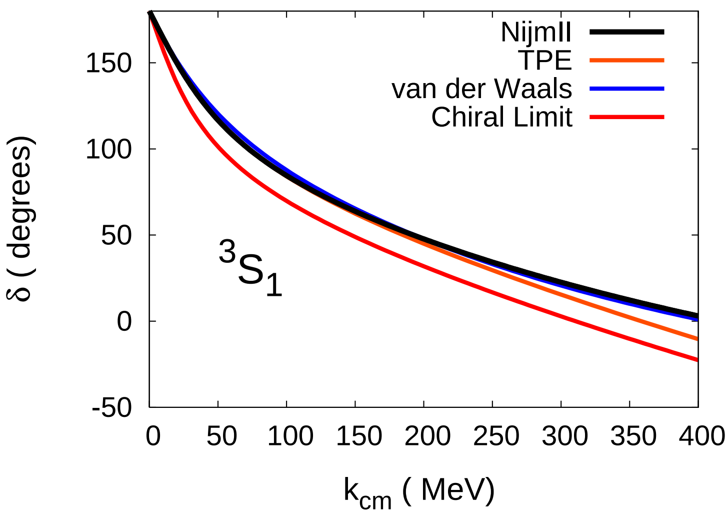
Dominance of Chiral Van der Waals forces in s-waves

Replace the s-wave potential by just the Van der Waals force and fix scattering lengths $\alpha_{0,s}$ and $\alpha_{0,t}$ to experiment $\rightarrow$ correlation between effective ranges independent of chiral constants.

Virial expansion $\rightarrow$ A Chiral Van der Waals nucleon gas ?







Renormalization Group (Independendence of short distances) Momentum space (Birse, McGovern , Richardson , 1997)

Change the boundary radius $R \to R + \Delta R$

$$\frac{\partial u(r,R)}{\partial R} = u_R(r,R) \tag{27}$$

Then,

$$\begin{aligned} u''(R,R) \; &+ \; u'_R(R,R) - L'_k(R) u(R,R) \\ &- \; L_k(R) \left(u'(R,R) + u_R(R,R) \right) = 0 \end{aligned} \tag{28}$$

Deriving also Schrödinger's equation with respect to the inner radius R we get

$$-u''_R(r,R) + U(r) u_R(r,R) = k^2 u_R(r,R) \tag{29}$$

and the asymptotic wave function

$$u_R(r,R) \to \sin(kr + \delta(k)) \tag{30}$$

we get

$$\begin{aligned} u_R(r,R) &\to \cos(kr + \delta(k)) \delta_R(k) \\ u'(r,R) &\to \cos(kr + \delta(k)) \\ u'_R(r,R) &\to -\sin(kr + \delta(k)) \delta_R(k) \end{aligned} \tag{31}$$

Thus, using Lagrange's identity we get

$$0 = -u_R u'' - u''_R u = (-u_R u' + u'_R u)' \tag{32}$$

Integrating between R and ∞ and using the boundary condition,

$$-k \frac{d\delta}{dR} = \left[k^2 - U(R) + L'_k(R) + L_k(R)^2 \right] u(R,R)^2 \tag{33}$$

This equation tells us how the phase shift changes as the inner radius is changed. We require the phase shift *not* to be dependent on R (renormalization group invariance) we get

$$-L'_k(R) = k^2 - U(R) + L_k(R)^2 \tag{34}$$

Fixed Points

$$L_k(R) = \frac{\xi_k(R)}{R} \tag{35}$$

The equation satisfied by $\xi(R)$ is

$$R\frac{d\xi_k}{dR} = \xi_k(1-\xi_k) + \left[U(R) - k^2\right] R^2 \tag{36}$$

Infrared (Large distance)

$$R\frac{d\xi_k}{dR} = \xi_k(1-\xi_k) \tag{37}$$

- $\xi_0 = 0$ Stable fixed point. Small scattering length
- $\xi_0 = 1$ Unstable fixed point. Large scattering length.

Ultraviolet (Short distance)

$$U(R) = \pm\frac{1}{a^2}\left(\frac{a}{R}\right)^n \qquad , \quad n \neq 2 \tag{38}$$

- For an attractive potential with $n < 2$ we have fixed points. This means that all solutions go to the irregular solution at the origin.
- If $n = 2$ and $g > -1/4$ we have fixed points. For $g < -1/4$ there are limit cycles.
- For $n > 2$ we have an atractor with fractal dimension $d = 1 - 2/n$.

These features are non-perturbative and are lost in perturbation theory.

OPE Renormalization of the Deuteron

Deuteron Equations

The Deuteron coupled channel $^3S_1 -^3 D_1$ equations

$$-u''(r) + U_s(r)u(r) + U_{sd}(r)w(r) = -\gamma^2 u(r)\,, \tag{39}$$

$$-w''(r) + U_{sd}(r)u(r) + \left[U_d(r) + \frac{6}{r^2}\right] w(r) = -\gamma^2 w(r)\,, \tag{40}$$

Asymptotic condition at infinity

$$\begin{aligned} u(r) &\to A_S e^{-\gamma r}\,, \\ w(r) &\to A_D e^{-\gamma r}\left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right)\,, \end{aligned} \tag{41}$$

$\gamma = \sqrt{MB}$ is the deuteron wave number
A_S is the normalization factor
$\eta = A_D/A_S$ asymptotic D/S ratio parameter

The OPE $^3S_1 -^3 D_1$ potential

$$U_s = U_c\,, \qquad U_{sd} = 2\sqrt{2}U_T\,, \qquad U_d = U_C - 2U_T\,, \tag{42}$$

OPE reduced potential ($U = 2\mu V$) is given for $r > 0$

$$U_C = -\frac{mMg_A^2}{16\pi f_\pi^2}\frac{e^{-mr}}{r} \tag{43}$$

$$U_T = -\frac{m^2Mg_A^2}{16\pi f_\pi^2}\frac{e^{-mr}}{r}\left(1 + \frac{3}{mr} + \frac{3}{(mr)^2}\right)\,, \tag{44}$$

We assume this potential to be valid for any strictly positive distance, $r \neq 0$, so the limit $r \to 0^+$ will be carefully taken, without subtracting any contribution.

Renormalization of singular potentials II. Coupled channel case (two)

Wave functions denoted by a column vector (u, w).

The reduced potential behaves as

$$U = M \frac{1}{r^n} \begin{pmatrix} C_S & C_E \\ C_E & C_D \end{pmatrix} \tag{45}$$

Diagonalizing the matrix

$$\begin{pmatrix} C_S & C_E \\ C_E & C_D \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $C_{n,\pm}$ are the corresponding eigenvalues and θ the mixing angle. Thus, at short distances we can decouple the equations to get

$$\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{46}$$

where (u_+, u_-) are regular solutions as in the single channel case.

Three cases

1. Both eigenvalues are negative.

   Two undetermined short distance phases, φ_+ and φ_-.

   Two states, (u_k, w_k) and (u_p, w_p), orthogonality constraint

$$\begin{aligned} 0 &= u_k' u_p - u_p u_k' + w_k' w_p - w_p w_k' \Big|_0 \\ &= \frac{1}{R_+} \sin\left(\varphi_+(k) - \varphi_+(p)\right) + \frac{1}{R_-} \sin\left(\varphi_-(k) - \varphi_-(p)\right) \end{aligned} \tag{47}$$

2. One eigenvalue is negative and the other is positive.

   One short distance phase φ.

   Two states (u_k, w_k) and (u_p, w_p) with different energies. Orthogonality constraint

$$\begin{aligned} 0 &= u_k' u_p - u_p u_k' + w_k' w_p - w_p w_k' \Big|_0 \\ &= \frac{1}{R_+} \sin\left(\varphi_+(k) - \varphi_+(p)\right) \end{aligned} \tag{48}$$

3. Both eigenvalues are negative.

   There are no short distance phases.

   The orthogonality relations are automatically satisfied.

The short distance regular solutions

We look for normalized functions,

$$1 = \int_0^\infty (u(r)^2 + w(r)^2) dr \,, \tag{49}$$

from which A_S can be determined.

At short distances the equations can be decoupled

$$\begin{aligned} u_A(r) &= \sqrt{\frac{2}{3}} u(r) + \frac{1}{\sqrt{3}} w(r) \,, \\ u_R(r) &= -\frac{1}{\sqrt{3}} u(r) + \sqrt{\frac{2}{3}} w(r) \,, \end{aligned} \tag{50}$$

yielding an attractive singular potential $U_A \to -4R/r^3$ for u_A and $U_R \to 8R/r^3$ for u_R.

$$\begin{aligned} u_R(r) &\to \left(\frac{r}{R}\right)^{3/4} \left[C_{1R} e^{+4\sqrt{2}\sqrt{\frac{R}{r}}} + C_{2R} e^{-4\sqrt{2}\sqrt{\frac{R}{r}}} \right] \,, \\ u_A(r) &\to \left(\frac{r}{R}\right)^{3/4} \left[C_{1A} e^{-4i\sqrt{\frac{R}{r}}} + C_{2A} e^{4i\sqrt{\frac{R}{r}}} \right] \,. \end{aligned} \tag{51}$$

The constants C_{1R}, C_{2R}, C_{1A} and C_{2A} depend on both γ and η and the OPE potential parameters, $g_{\pi NN}$ and m.

The normalizability of the wave function at the origin requires

$$C_{1R}(\gamma, \eta) = 0 \,, \tag{52}$$

Thus, in all one has three independent variables

- The binding energy, or γ
- The OPE coupling constant
- The pion mass

Then η is predicted.

In summary

$$\begin{aligned} u_R(r) &\to C_R(\gamma) \left(\frac{r}{R}\right)^{3/4} e^{-4\sqrt{2}\sqrt{\frac{R}{r}}} \,, \\ u_A(r) &\to C_A(\gamma) \left(\frac{r}{R}\right)^{3/4} \sin\left[4\sqrt{\frac{R}{r}} + \varphi(\gamma) \right] \,. \end{aligned} \tag{53}$$

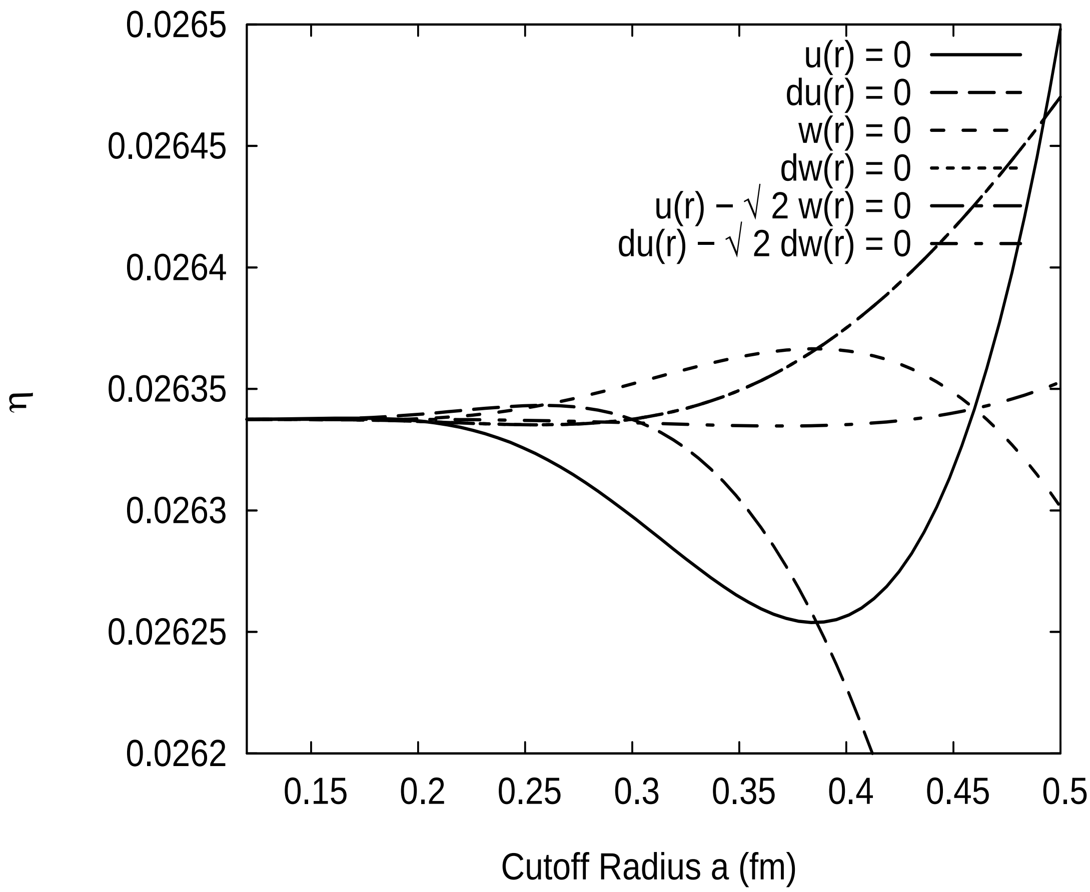
Regularization with boundary conditions

Short distance cut-off

$$
\begin{aligned}
u(a) &= 0 \qquad (\mathrm{BC1})\,, \\
u'(a) &= 0 \qquad (\mathrm{BC2})\,, \\
w(a) &= 0 \qquad (\mathrm{BC3})\,, \\
w'(a) &= 0 \qquad (\mathrm{BC4})\,, \\
u(a) - \sqrt{2}w(a) &= 0 \qquad (\mathrm{BC5})\,, \\
u'(a) - \sqrt{2}w'(a) &= 0 \qquad (\mathrm{BC6})\,,
\end{aligned}
\tag{54}
$$

Superposition principle of boundary conditions

$$
\begin{aligned}
u(r) &= u_S(r) + \eta\, u_D(r) \\
w(r) &= w_S(r) + \eta\, w_D(r)\,,
\end{aligned}
\tag{55}
$$



Deuteron properties

Matter radius

$$r_m^2 = \frac{1}{4}\langle r^2 \rangle = \frac{1}{4}\int_0^\infty r^2 (u(r)^2 + w(r)^2) dr \tag{56}$$

Quadrupole moment (without meson exchange currents)

$$Q_d = \frac{1}{20}\int_0^\infty r^2 w(r)(2\sqrt{2}u(r) - w(r)) dr \tag{57}$$

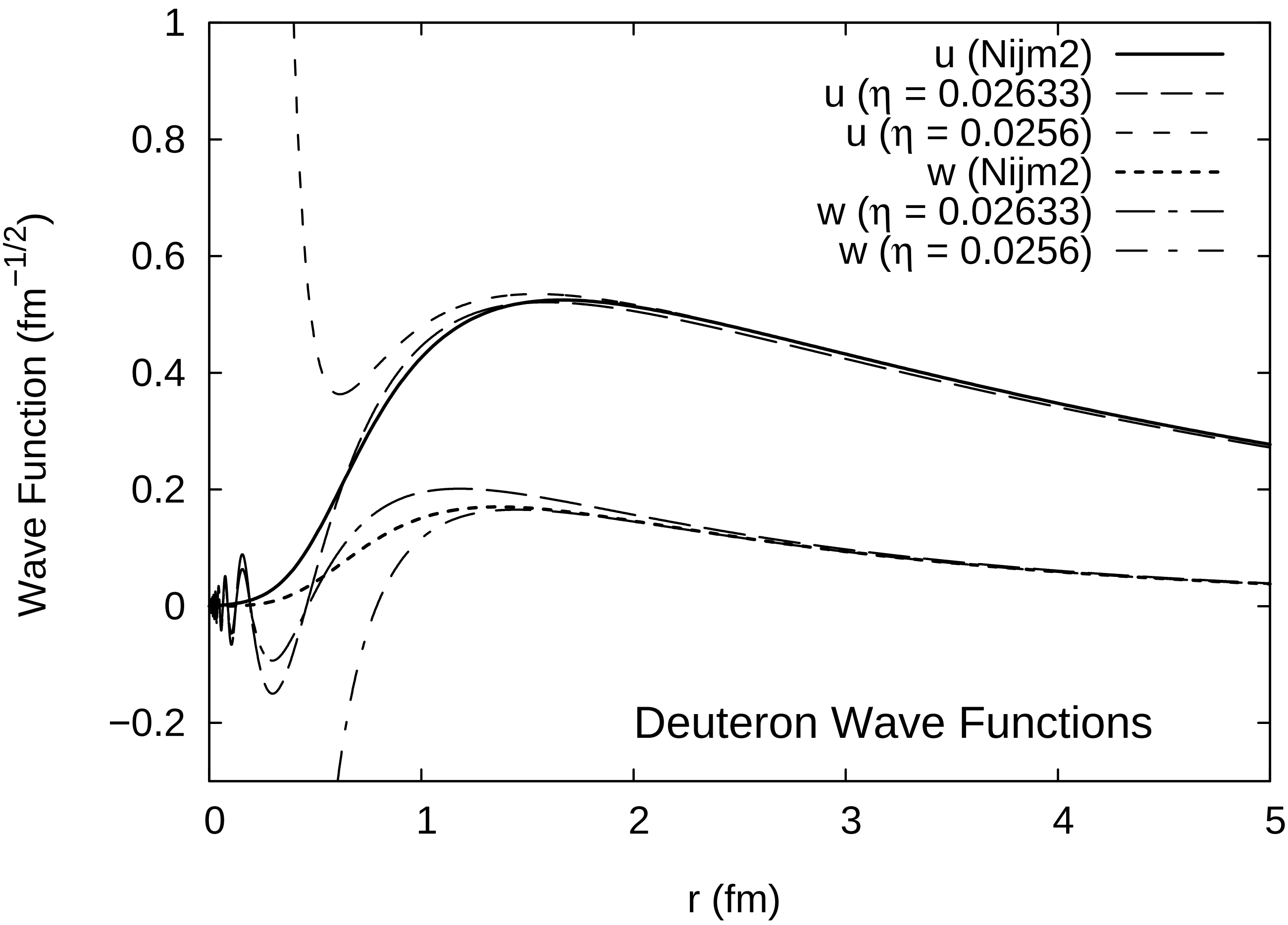
Deuteron inverse radius

$$\langle r^{-1} \rangle = \int_0^\infty dr \frac{u(r)^2 + w(r)^2}{r} \tag{58}$$

D-state probability is given by

$$P_D = \int_0^\infty w(r)^2 dr \tag{59}$$

Wave functions



OPE Correlations in deuteron observables

Weak binding limit

$$\eta^{\rm OPE} = 0.9638\gamma^2 - 3.46864\gamma^3 + \mathcal{O}(\gamma^4) \tag{60}$$
$$\frac{A_S^{\rm OPE}}{\sqrt{2\gamma}} = 1 + 1.2455\gamma - 0.4705\gamma^2 + \mathcal{O}(\gamma^3) \tag{61}$$
$$\sqrt{8}\,\gamma\; r_m^{\rm OPE} = 1 + 1.2455\gamma - 0.4705\gamma^2 + \mathcal{O}(\gamma^3) \tag{62}$$
$$Q_d^{\rm OPE} = 0.6815 - 3.5437\gamma + \mathcal{O}(\gamma^2) \tag{63}$$

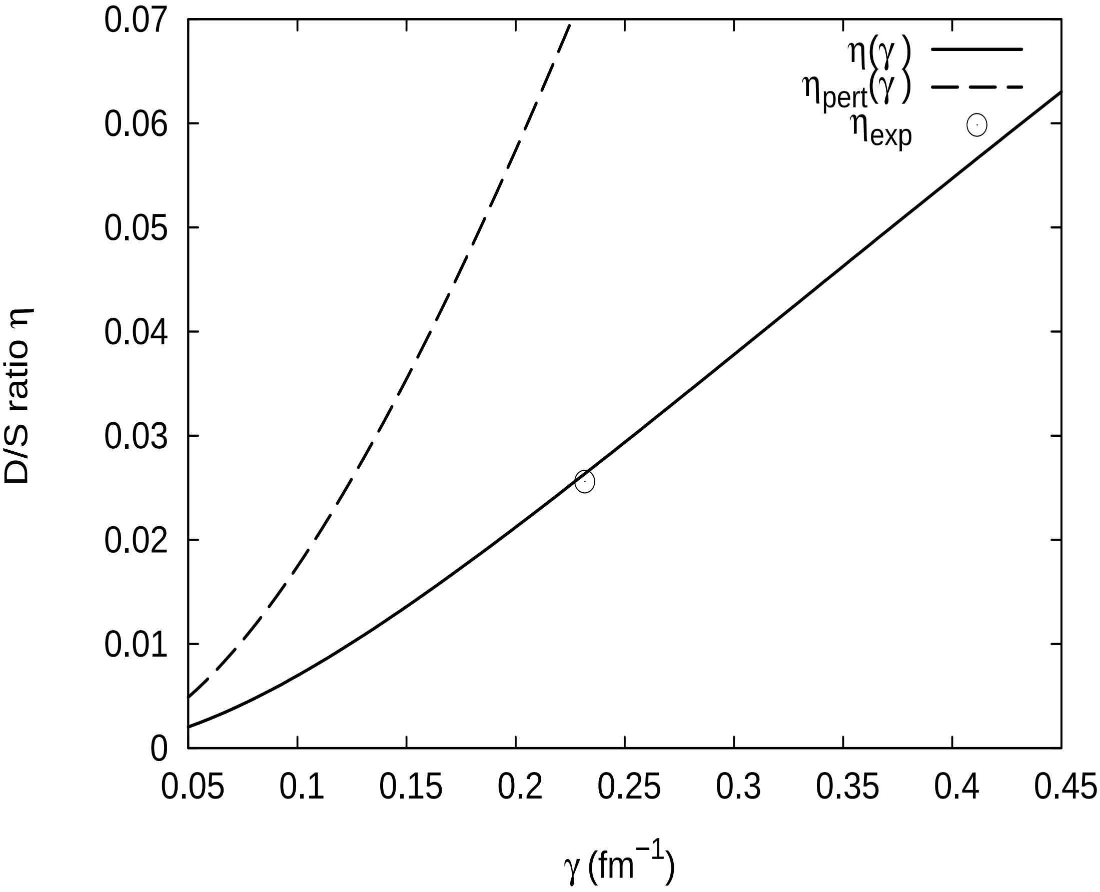
Weak binding correlations

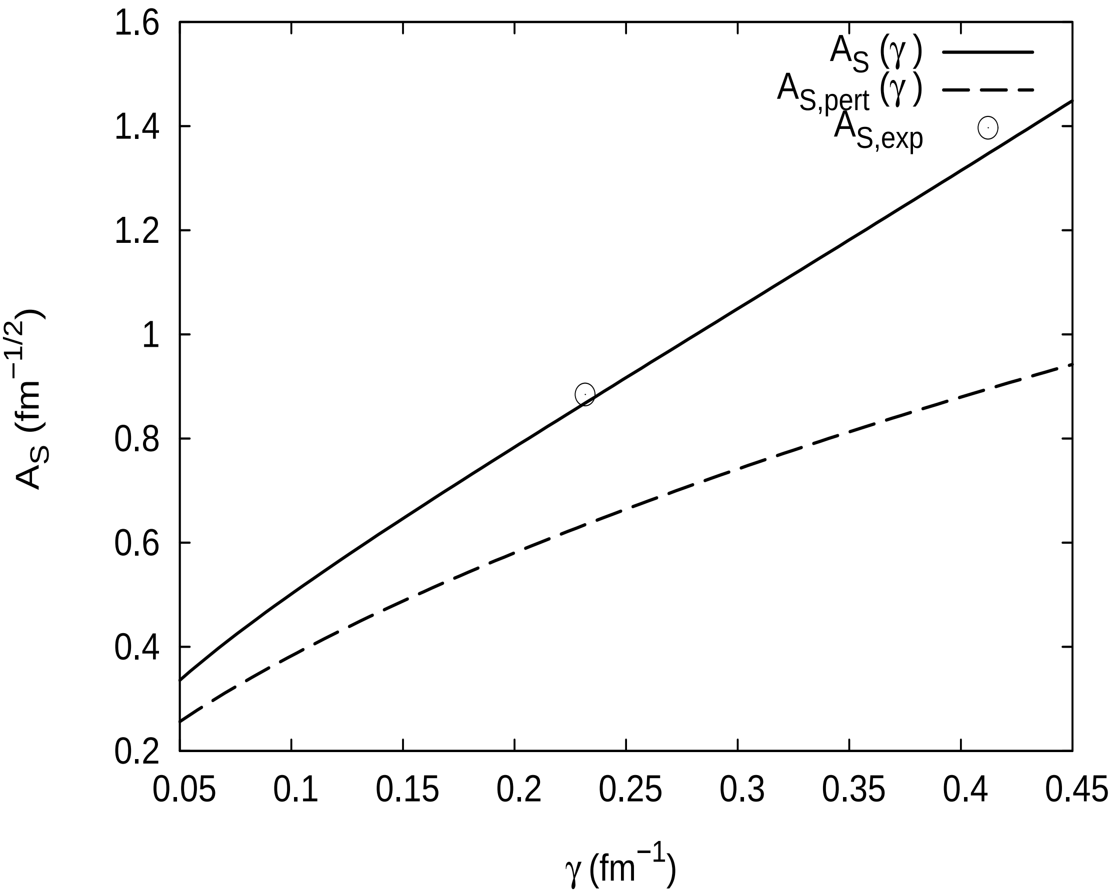
$$r_m = \frac{A_S}{4\,\gamma^{3/2}} + \mathcal{O}(\gamma^3) \tag{64}$$

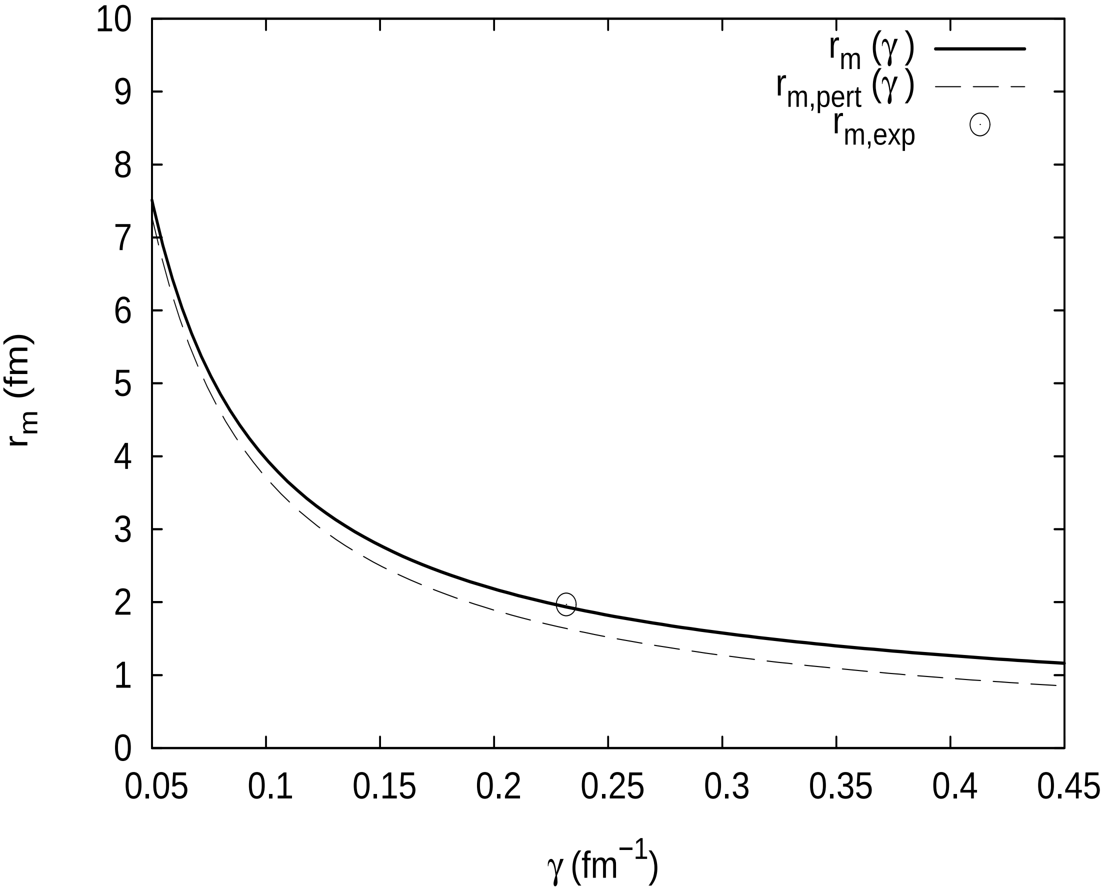
$$\frac{\sqrt{2}\gamma^2 Q_d}{\eta_d} = 1 + \mathcal{O}(\gamma) \tag{65}$$

Comparison with perturbation theory

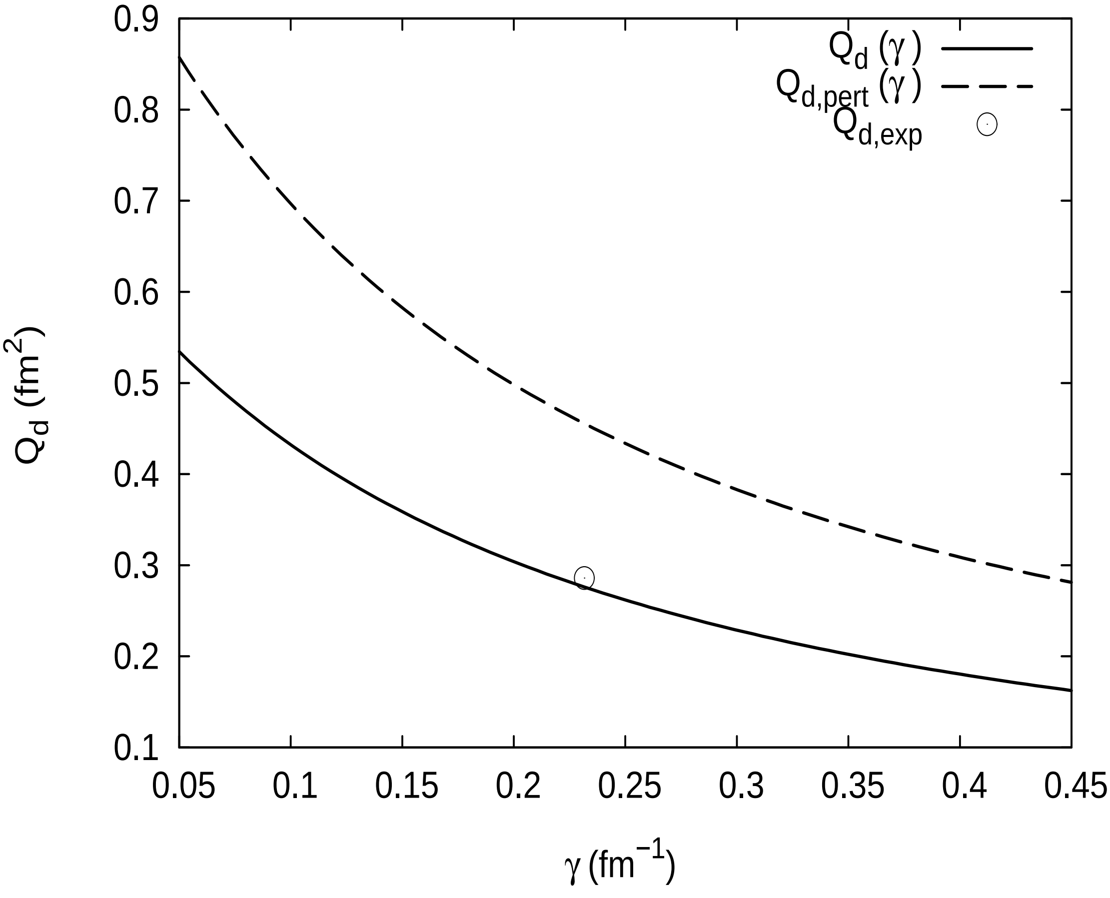
$$\begin{aligned}
\eta_{\rm pert} &= 1.5497\gamma^2 - 4.15479\gamma^3 + \mathcal{O}(\gamma^4, R^2) \\
\frac{A_{S,\rm pert}}{\sqrt{2\gamma}} &= 1 - 0.7184\gamma - 2.7394\gamma^2 + \mathcal{O}(\gamma^3, R^2) \\
r_{m,\rm pert}\sqrt{8}\gamma &= 1 + 0.71843\gamma - 2.7394\gamma^2 + \mathcal{O}(\gamma^3, R^2) \\
Q_{\rm pert} &= 1.09587 - 5.87576\gamma + \mathcal{O}(\gamma^2, R^2)
\end{aligned} \tag{66}$$


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
D/S ratio η
γ (fm^{-1})
η(γ)
η_{pert}(γ)
η_{exp}


1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
A_S (fm$^{-1/2}$)
γ (fm^{-1})
A_S (γ)
$A_{S,pert}$ (γ)
$A_{S,exp}$



r_m (fm)
γ (fm^-1)
r_m (γ)
r_m,pert (γ)
r_m,exp



Q_d (fm^2)
γ (fm^-1)
Q_d (γ)
Q_d,pert (γ)
Q_d,exp

Scattering properties in the $^3S_1 - {}^3D_1$ channel

α and β scattering states normalization

$$
\begin{aligned}
u_{k,\alpha}(r) &\to \frac{\cos\epsilon}{\sin\delta_1}\Big(\hat{j}_0(kr)\cos\delta_1 - \hat{y}_0(kr)\sin\delta_1\Big)\,, \\
w_{k,\alpha}(r) &\to \frac{\sin\epsilon}{\sin\delta_1}\Big(\hat{j}_2(kr) - \hat{y}_2(kr)\sin\delta_1\Big)\,, \\
u_{k,\beta}(r) &\to -\frac{1}{\sin\delta_1}\Big(\hat{j}_0(kr)\cos\delta_2 - y_0(kr)\sin\delta_2\Big)\,, \\
w_{k,\beta}(r) &\to \frac{\tan\epsilon}{\sin\delta_1}\Big(\hat{j}_2(kr)\cos\delta_2 - \hat{y}_2(kr)\sin\delta_2\Big)\,,
\end{aligned}
$$

Short distance behaviour

$$
\begin{aligned}
u_{R,\alpha}(r) &\to C_{R,\alpha}(\gamma)\left(\frac{r}{R}\right)^{3/4} e^{-4\sqrt{2}\sqrt{\frac{R}{r}}}\,, \\
u_{A,\alpha}(r) &\to C_{A,\alpha}(\gamma)\left(\frac{r}{R}\right)^{3/4} \sin\left[4\sqrt{\frac{R}{r}} + \varphi_\alpha\right]\,.
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
u_{R,\beta}(r) &\to C_{R,\beta}(\gamma)\left(\frac{r}{R}\right)^{3/4} e^{-4\sqrt{2}\sqrt{\frac{R}{r}}}\,, \\
u_{A,\beta}(r) &\to C_{A,\beta}(\gamma)\left(\frac{r}{R}\right)^{3/4} \sin\left[4\sqrt{\frac{R}{r}} + \varphi_\beta\right]\,.
\end{aligned}
\tag{68}
$$

Orthogonality constraints for singular potentials

$$
\begin{aligned}
0 &= (\gamma^2 + k^2)\int_0^\infty dr\Big[u_\gamma(r)u_k(r) + w_\gamma(r)w_k(r)\Big] \\
&= \Big[u'_\gamma u_k - u_\gamma u'_k + w'_\gamma w_k - w_\gamma w'_k\Big]\Big|_0^\infty
\end{aligned}
\tag{69}
$$

Then

$$
C_{A,i}(k)C_A(\gamma)\sin\left[\varphi(\gamma) - \varphi_i(k)\right] = 0 \qquad , i = \alpha,\beta \tag{70}
$$

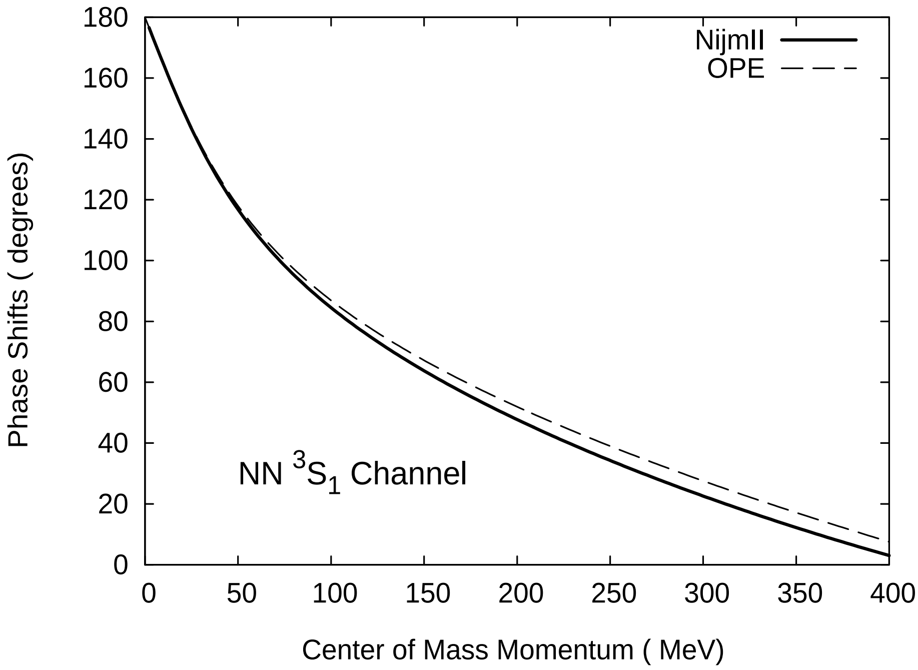
and

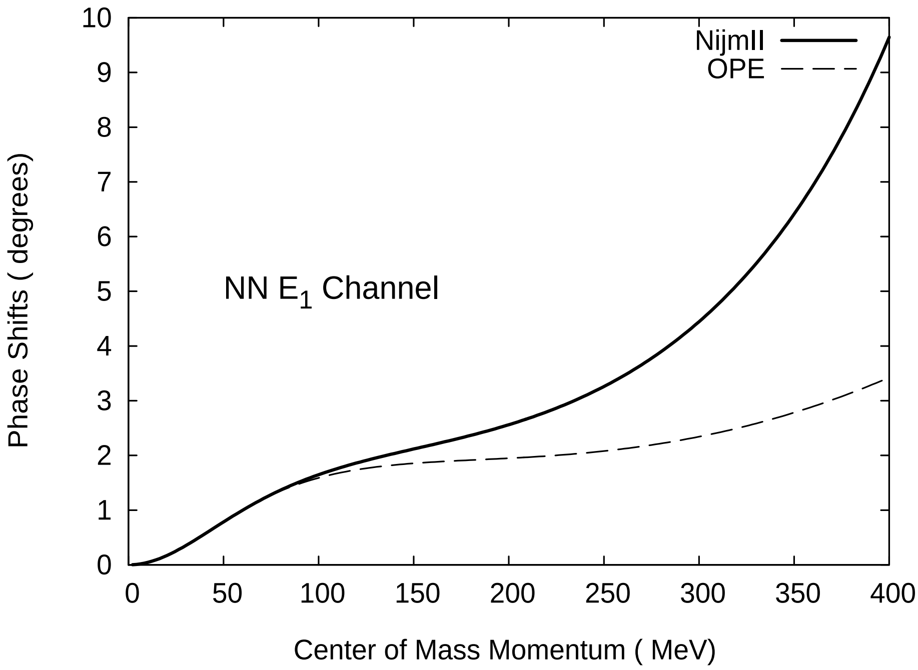
$$
\varphi(\gamma) = \varphi_\alpha(k) = \varphi_\beta(k) \tag{71}
$$

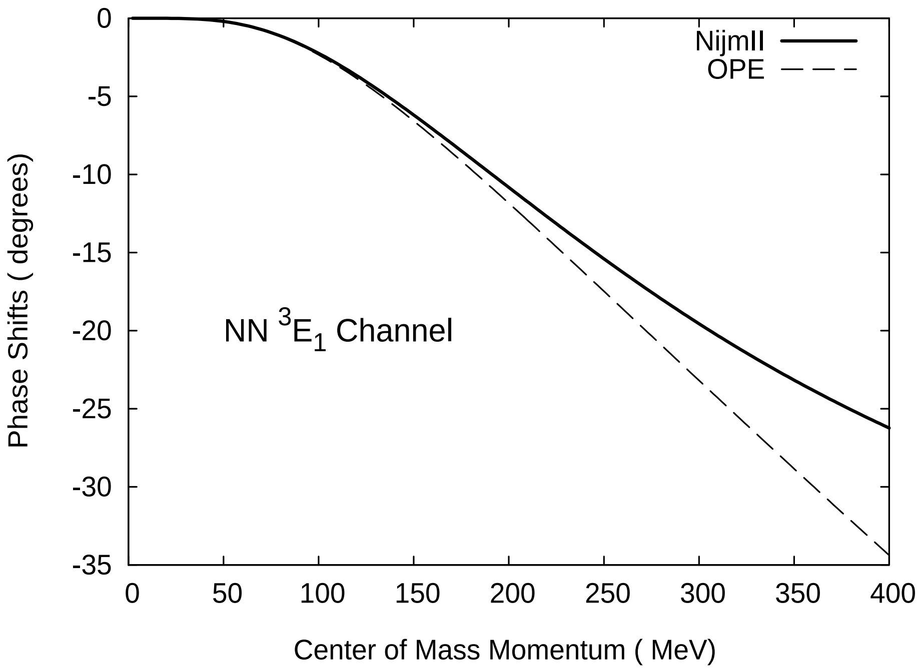
There is only one short distance phase.

Phase shifts are predicted.

Perturbation theory of Kaplan,Savage and Wise violates orthogonality perturbatively.


NijmII
OPE
NN $^3S_1$ Channel
Phase Shifts (degrees)
Center of Mass Momentum (MeV)
0
20
40
60
80
100
120
140
160
180
0
50
100
150
200
250
300
350
400


NijmII
OPE
NN E_1 Channel
Phase Shifts (degrees)
Center of Mass Momentum (MeV)
0
1
2
3
4
5
6
7
8
9
10
0
50
100
150
200
250
300
350
400


NijmII
OPE
NN $^3E_1$ Channel
Phase Shifts (degrees)
Center of Mass Momentum (MeV)
0
-5
-10
-15
-20
-25
-30
-35
0
50
100
150
200
250
300
350
400

Low energy parameters

In the low energy limit one has

$$\begin{aligned} \delta_1 &\rightarrow -\alpha_0 k\,, \\ \delta_2 &\rightarrow -\alpha_2 k^5\,, \\ \epsilon &\rightarrow \frac{\alpha_{02}}{\alpha_0}k^2 \end{aligned} \tag{72}$$

so that the zero energy the wave functions behave asymptotically

$$\begin{aligned} u_{0,\alpha}(r) &\rightarrow 1-\frac{r}{\alpha_0}\,, \\ w_{0,\alpha}(r) &\rightarrow \frac{3\alpha_{02}}{\alpha_0 r^2}\,, \\ u_{0,\beta}(r) &\rightarrow \frac{r}{\alpha_0}\,, \\ w_{0,\beta}(r) &= \frac{3\alpha_2}{\alpha_{02}r^2}-\frac{r^3}{15\alpha_{02}}\,. \end{aligned} \tag{73}$$

Superposition principle of boundary conditions

$$\begin{aligned} u_{0,\alpha}(r) &= u_1(r)-\frac{1}{\alpha_0}u_2(r)+\frac{3\alpha_{02}}{\alpha_0}u_3(r) \\ w_{0,\alpha}(r) &= w_1(r)-\frac{1}{\alpha_0}w_2(r)+\frac{3\alpha_{02}}{\alpha_0}w_3(r) \\ u_\beta(r) &= \frac{1}{\alpha_0}u_2(r)+\frac{3\alpha_2}{\alpha_{02}}u_3(r)-\frac{1}{15\alpha_{02}}u_4(r) \\ w_\beta(r) &= \frac{1}{\alpha_0}w_2(r)+\frac{3\alpha_2}{\alpha_{02}}w_3(r)-\frac{1}{15\alpha_{02}}w_4(r) \end{aligned}$$

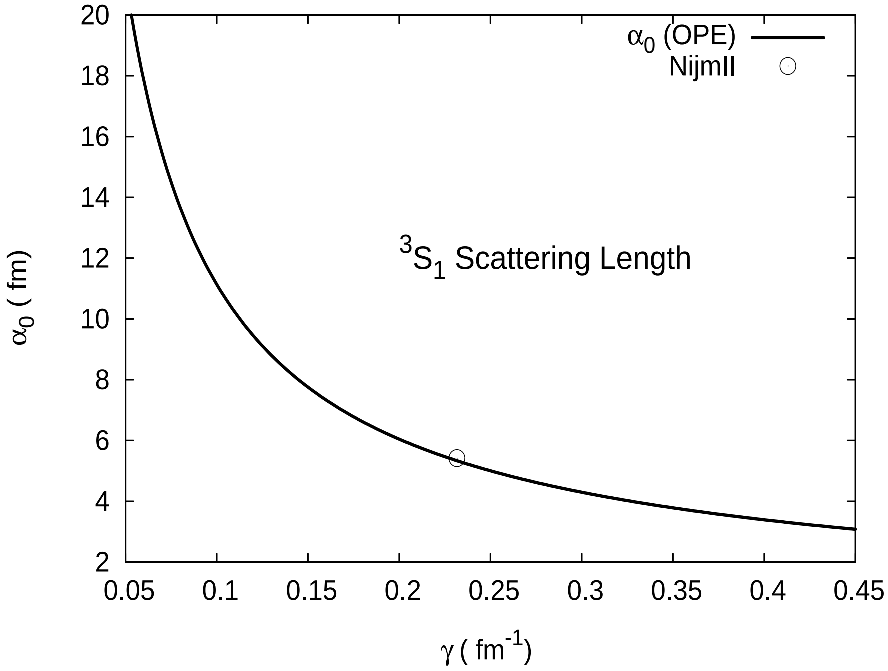
$u_{1,2,3,4}$ and $w_{1,2,3,4}$ are independent on α_0, α_{02} and α_2

Regularity Correlations

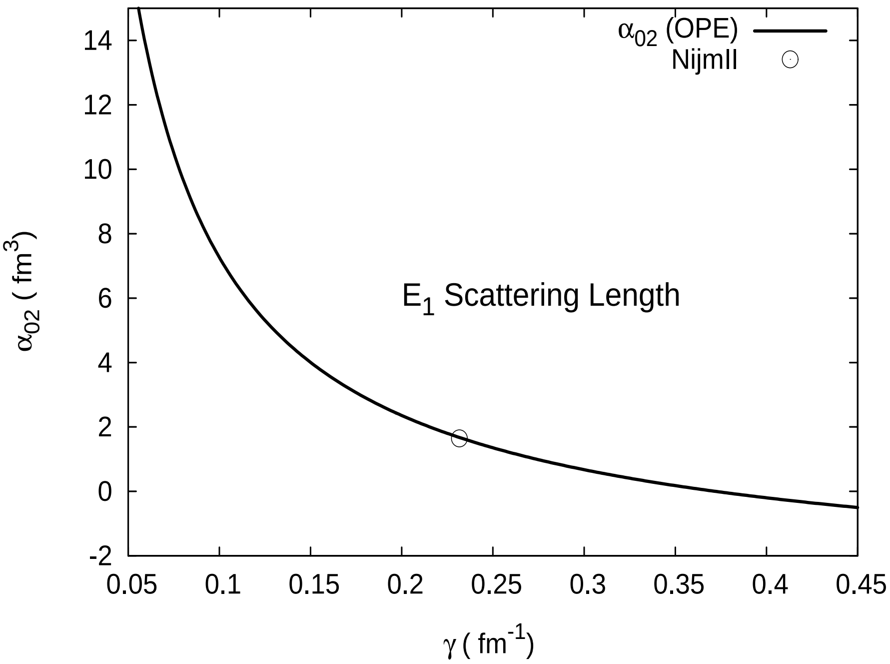
$$\begin{aligned} \alpha_{02} &= 0.963571370240\,\alpha_0-3.467616391389 \\ \alpha_2 &= 3.467616391389\,\frac{\alpha_{02}}{\alpha_0}+5.080264230656 \end{aligned} \tag{74}$$

Orthogonality Correlations

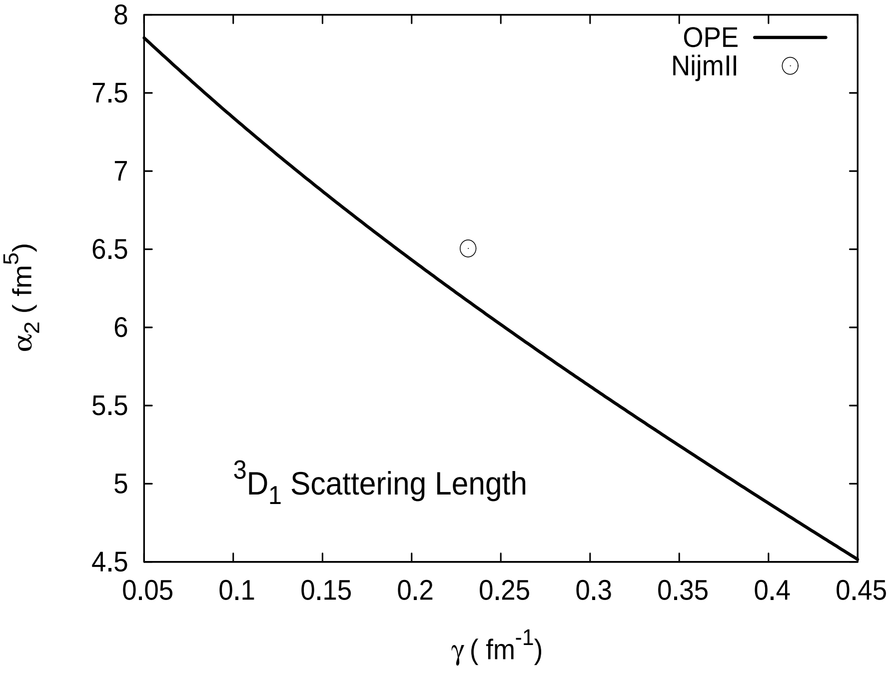
$$\begin{aligned} \alpha_0 &= 1.037805911852\,\alpha_{02}+3.598712446758 && (\alpha) \\ \alpha_{02} &= 0.288382561043\,\alpha_0\,\alpha_2-1.465059639612\,\alpha_0 && (\beta) \end{aligned} \tag{75}$$



α_0 (OPE)
NijmII
$^3S_1$ Scattering Length
α_0 (fm)
γ (fm^{-1})



α_{02} (OPE)
NijmII
E_1 Scattering Length
α_{02} (fm^3)
γ (fm^{-1})



OPE
NijmII
$^3D_1$ Scattering Length
α_2 (fm^5)
γ (fm^{-1})

Chiral TPE Renormalization of the Deuteron

- Attractive-Attracttive singular eigenpotentials

$$U = M\frac{1}{r^6}\begin{pmatrix} C_S & C_E \\ C_E & C_D \end{pmatrix} < 0$$

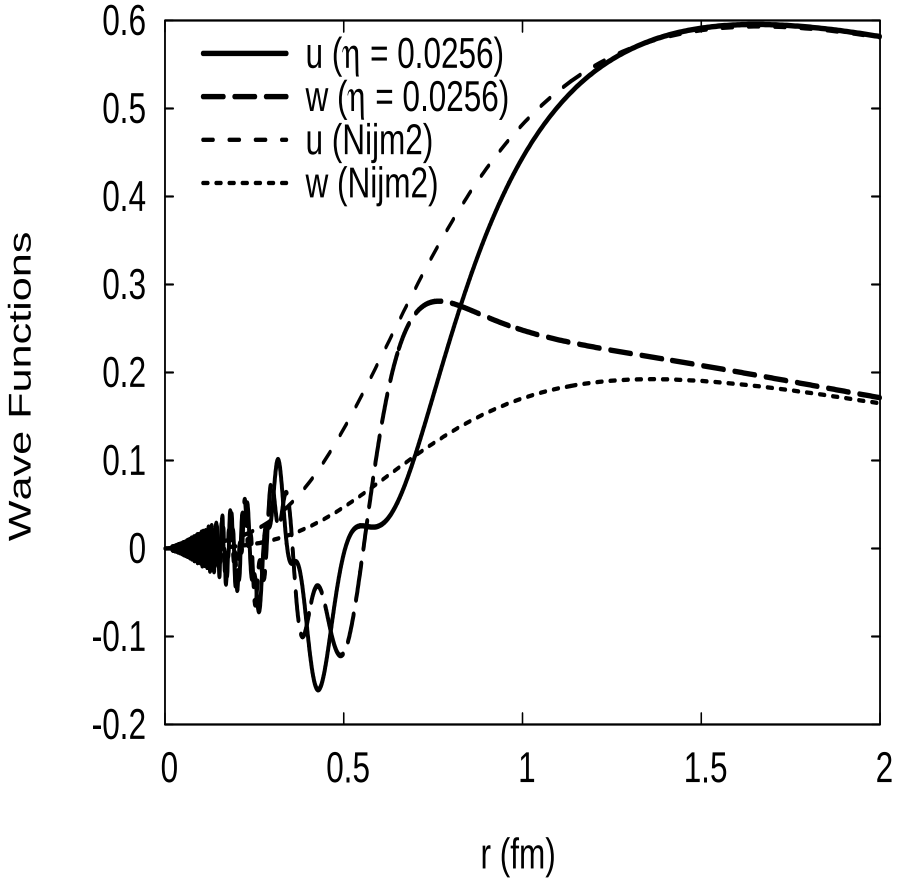
3 independent parameters for a given energy

- For zero energy scattering → Scattering length matrix

$$\begin{pmatrix} \alpha_0 & \alpha_{02} \\ \alpha_{02} & \alpha_2 \end{pmatrix}$$

For the deuteron γ, η and A_S but normalization

- Orthogonality between zero energy state and bound state → α_0, γ and η as independent parameters.
- Orthogonality to positive energy states → Phase-shifts in the triplet $^3S_1 - ^3D_1$ channel deduced



Predictive power ?

MonteCarlo Error propagation of all inputs $g_{\pi NN} = 13.1 \pm 0.1$, $\alpha_{0,s} = -23.77 \pm 0.05$, $\alpha_{0,t} = 5.419 \pm 0.007$, $\eta = 0.0256 \pm 0.0004$ and the chiral constants c_1, c_3 and c_4.

	Set I πN	Set II Rentmeester'99	Set IV Machleidt'03	Exp.
c_1	$-0.81(15)$	$-0.76(7)$	-0.81	
c_3	$-4.69(1.34)$	$-5.08(24)$	$-3.20(16)$	
c_4	$3.40(4)$	$4.78(10)$	$5.40(1.65)$	
$r_{0,s}$	$2.92^{+0.08}_{-0.04}$	$2.97^{+0.03}_{-0.02}$	$2.86^{+0.04}_{-0.03}$	2.77 ± 0.05
$r_{0,t}$	$1.36^{+0.33}_{-0.75}$	$1.48^{+0.14}_{-0.25}$	$1.76^{+0.03}_{-0.06}$	1.753 ± 0.008
A_s	$0.899^{+0.008}_{-0.009}$	$0.900^{+0.003}_{-0.004}$	$0.884^{+0.005}_{-0.008}$	0.8849 ± 0.0009
Q_d	$0.284^{+0.005}_{-0.007}$	$0.284^{+0.005}_{-0.004}$	$0.276^{+0.004}_{-0.004}$	0.2859 ± 0.0003
r_m	$1.998^{+0.015}_{-0.019}$	$1.998^{+0.007}_{-0.007}$	$1.965^{+0.011}_{-0.014}$	1.971 ± 0.006
P_d	$6.6^{1.0}_{-0.9}$	$7.1^{+0.9}_{-0.9}$	$8.3^{+1.4}_{-1.5}$	–
α_{02}	$2.26^{+0.51}_{-0.39}$	$2.20^{+0.23}_{-0.16}$	$1.67^{+0.13}_{-0.13}$	–
α_2	$2.6^{+2.8}_{-6.6}$	$3.1^{+1.4}_{-2.8}$	$6.17^{+0.39}_{-0.75}$	–

- Does it make sense to compute higher orders ?
- MEC's to the quadrupole moment ?

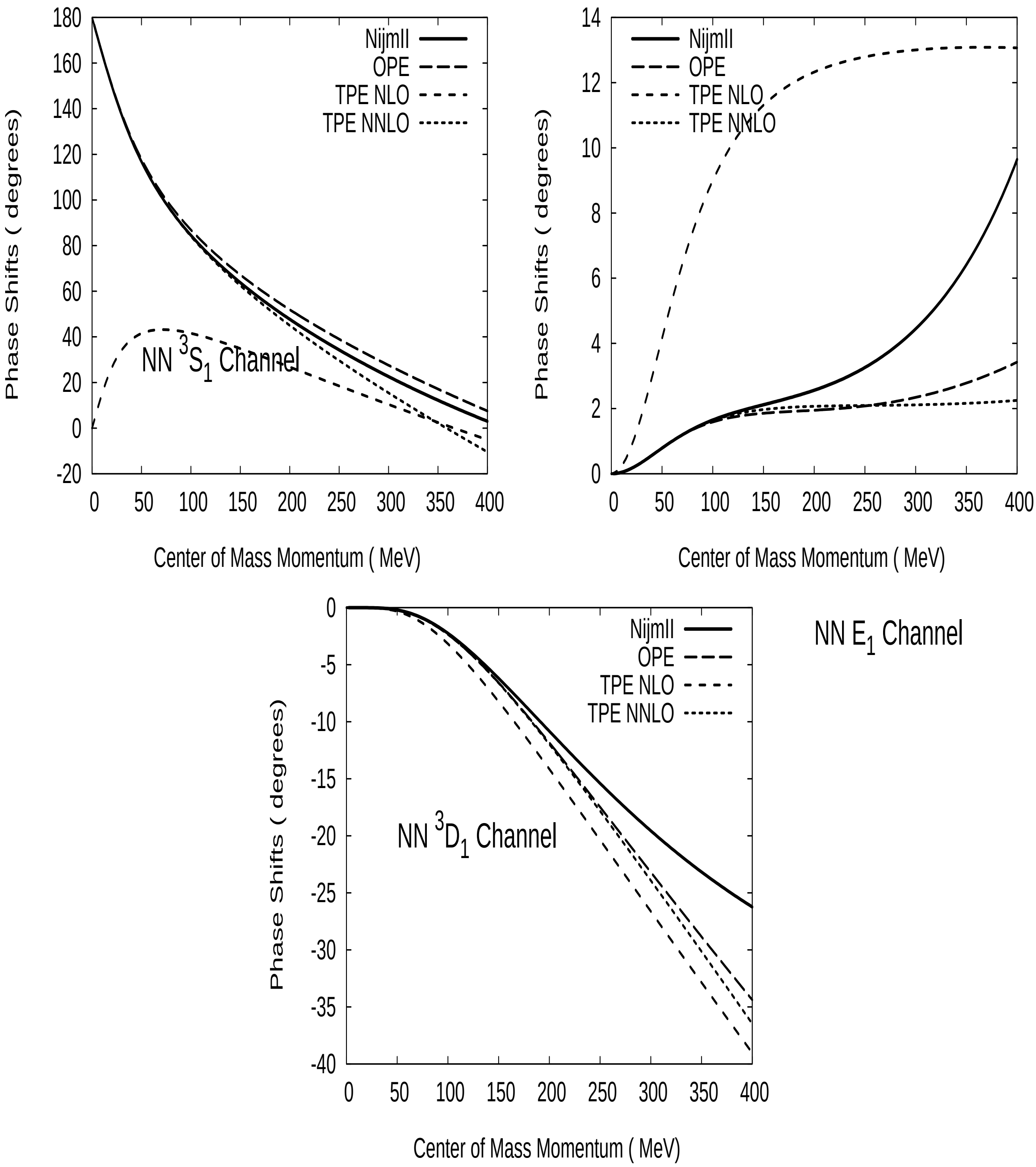
Further details → Talk of Manolo Pavón in this Workshop

The NLO problem in the triplet channel: A missing link

In the triplet channel we have

$$U_{\rm NLO} = M \frac{1}{r^5} \begin{pmatrix} C_S & C_E \\ C_E & C_D \end{pmatrix} > 0$$

No free parameters !!



- BUT using relativistic potentials (Higa 2005) one gets relativistic Van der Waals

$$U_{\rm TPE} = \frac{M}{r^7} \begin{pmatrix} C_S & C_E \\ C_E & C_D \end{pmatrix} = U_{\rm NLO} + U_{\rm NNLO} + \dots \tag{76}$$

- In the heavy baryon expansion we get different leading short distance behaviour because Mr enters and $M \to \infty$ and $r \to 0$ are conflicting limits
- For the relativistic problem we have attractive-repulsive eigenpotentials which means ONE free parameters as in the OPE case.
- Obvious candidate for trying out $\to$ detailed investigation

Requirements on Power Counting for singular potentials

- Power counting on the potentials is well defined

$$
\begin{aligned}
U(r) = & \frac{M_N m_\pi^3}{f_\pi^2} W_{\rm LO}(m_\pi r, g_A) + \frac{M_N m_\pi^5}{f_\pi^4} W_{\rm NLO}(m_\pi r, g_A) \\
& + \frac{m_\pi^6}{f_\pi^4} W_{\rm NNLO}(m_\pi r, g_A, c_1, c_3, c_4) + \dots
\end{aligned}
$$

- Nogga, Timmermans, Van Kolck (2005) propose to promote counterterms which are infrared enchanced, because some phase-shifts depend strongly on the cut-off.
- The promotable counterterms can never be anticipated unless the detailed short distance behaviour of the potential is determined. The power counting in the potential did not know how was the singularity or if the singularity existed at all.
- We propose also selection rules : ultraviolet enhanced counterterms should be demoted for the same reasons.
- The short distance singularity of the long distance potential constraints the form of the counterterms if you want a finite limit but there is no need to violate the counting if you keep a finite cut-off (which one ?).

Conclusions

- Singular potentials do have a meaning. They are renormalizable non-perturbatively. They become perturbatively non-renormalizable. That means that short distance ambiguities are controlable if appropriate boundary conditions are adopted.
- The deuteron equations can be renormalized in OPE and TPE potentials
- OPE parameters and binding energy only inputs
- TPE parameters, binding energy, scattering length and mixing only inputs
- All other deuteron properties agree within few percent accuracy
- Orthogonality constraints allow to predict also scattering observables
- Short distance contributions to deuteron and scattering observables are forbidden due to consistency constraints