Variational Description of Bound and Scattering States in Three- and Four-Nucleon Systems

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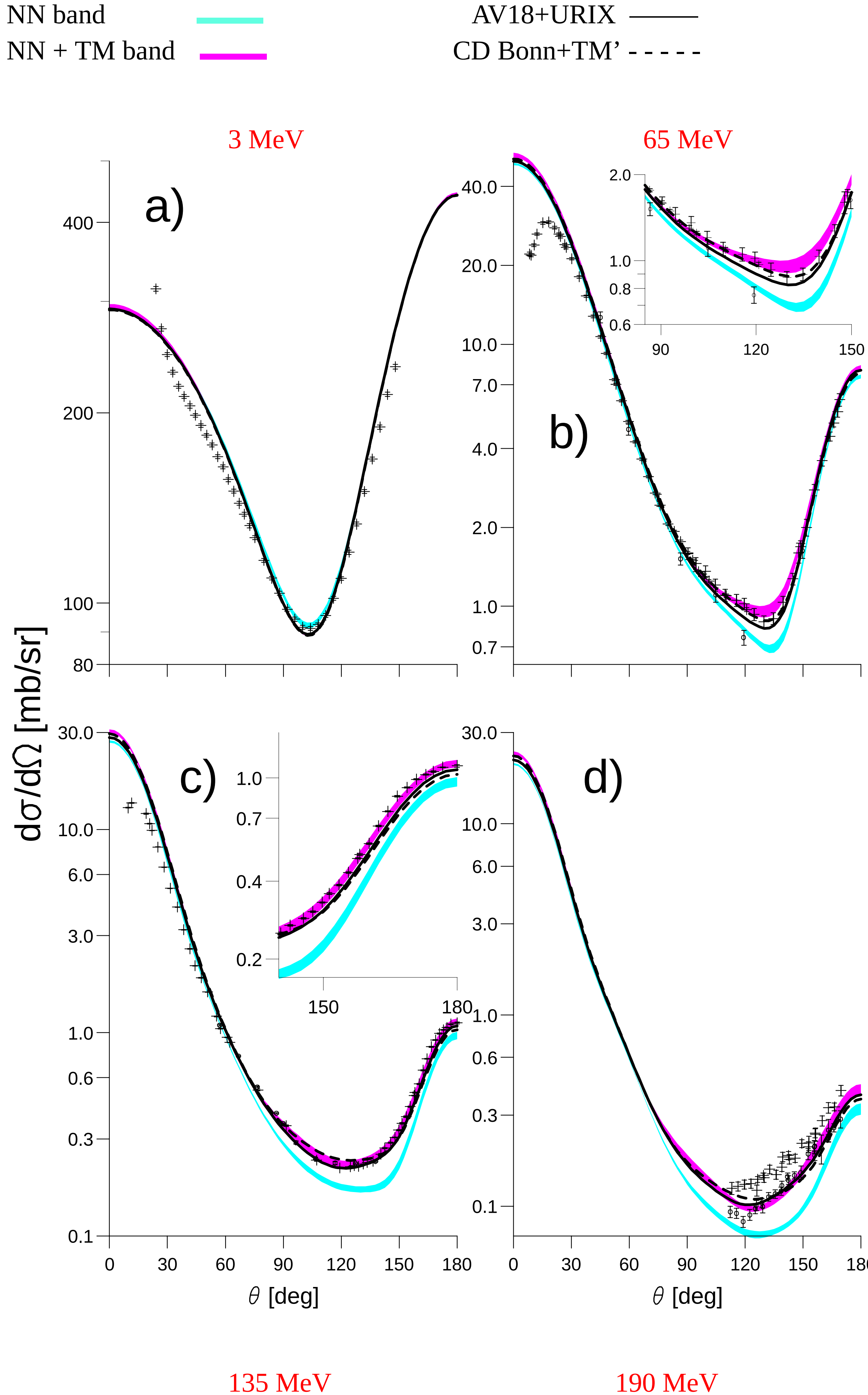
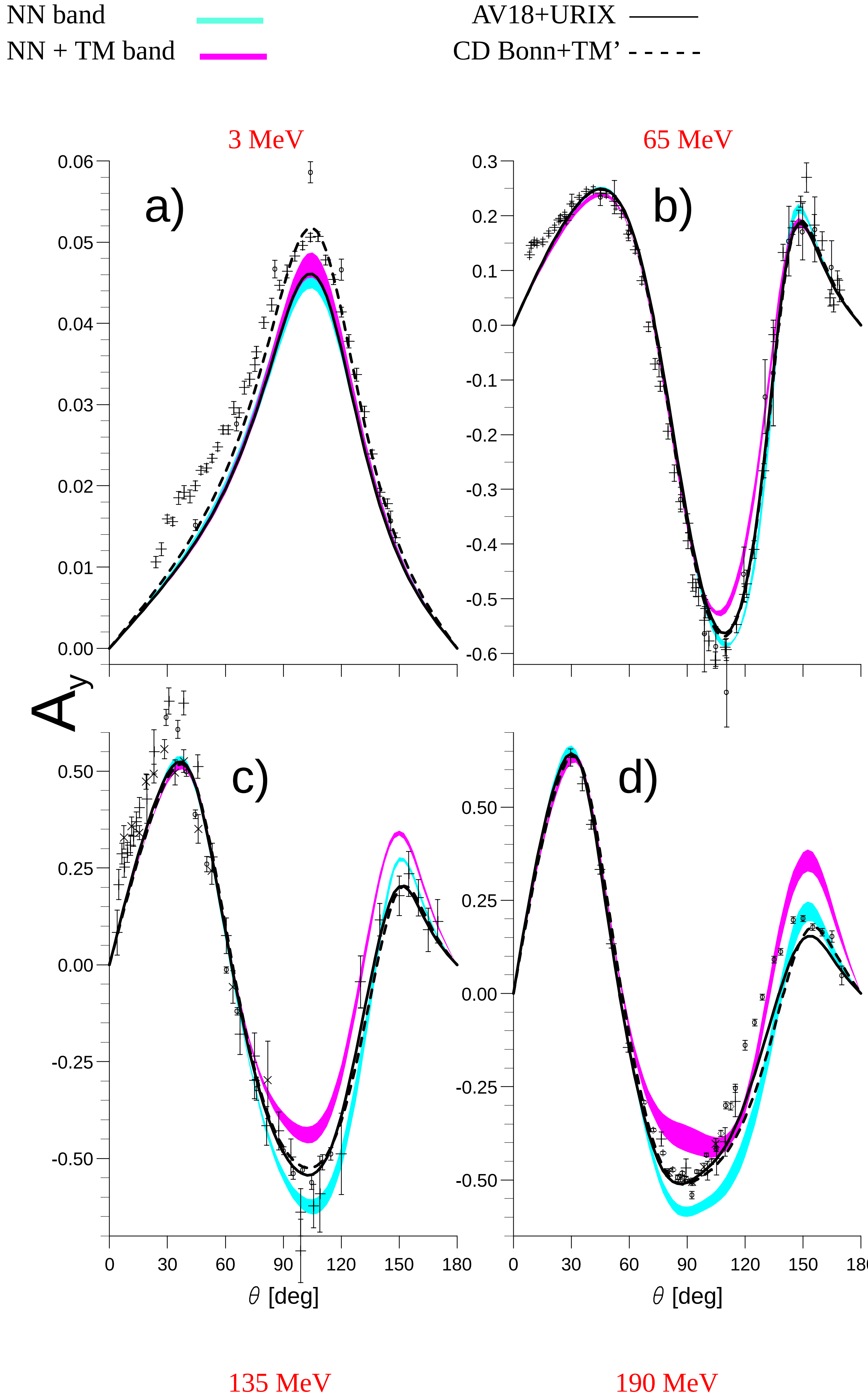
Motivation

- Realistic NN potentials describe 2N data with $\chi^2 \approx 1$
- Realistic NN potentials describe 3N data with $\chi^2 >> 1$
- Realistic NN+3N potentials describe 3N data with $\chi^2 >> 1$

Topics

- Construction of $A = 3, 4$ bound state wave functions
- Construction of three- and four-nucleon scattering states
- Production of benchmarks:
- a) bound states
- b) scattering states
- Calculations of observables
- Theory vs experiment
- $n - d,\ \ p - d,\ \ n - {}^3\mathrm{H},\ \ p - {}^3\mathrm{He}$ elastic scattering
- capture reactions as $p + d \rightarrow {}^3\mathrm{He} + \gamma$ or $n + d \rightarrow {}^3\mathrm{H} + \gamma$

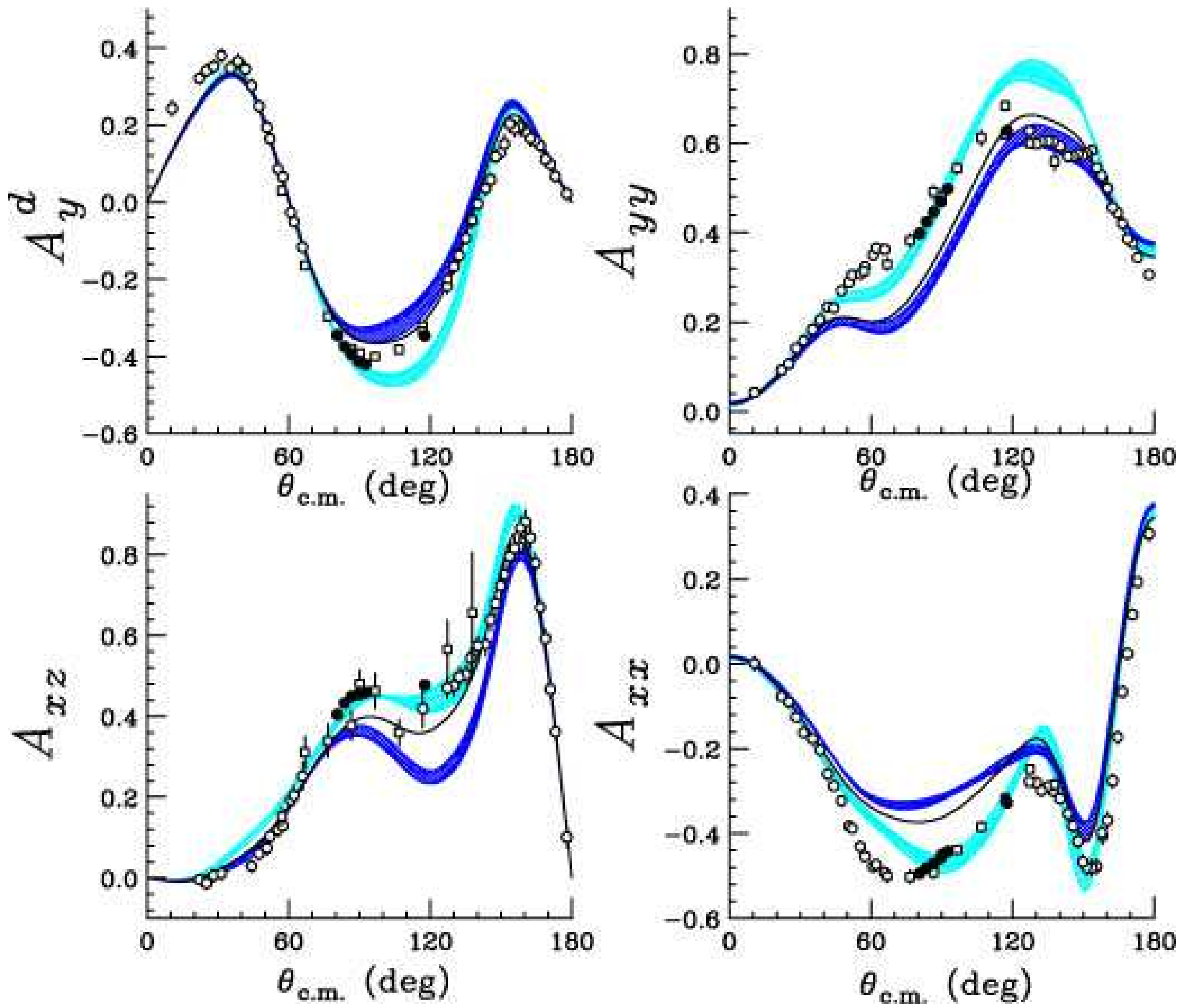

NN band
NN + TM band
AV18+URIX
CD Bonn+TM'
3 MeV
65 MeV
a)
b)
c)
d)
dσ/dΩ [mb/sr]
θ [deg]
135 MeV
190 MeV



NN band
NN + TM band
AV18+URIX
CD Bonn+TM'
3 MeV
65 MeV
a)
b)
c)
d)
A_y
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0.50
0.25
0.00
-0.25
-0.50
0
30
60
90
120
150
180
θ [deg]
135 MeV
190 MeV

Polarization observables measured at RIKEN at 135 MeV

NN band ▬▬ NN + TM band ▬▬ AV18+URIX ——



$A = 3, 4$ bound states

The A=3,4 wave function is written as

$$\Psi_3 = \sum_{i=1,3} \psi(\mathbf{x}_i, \mathbf{y}_i)$$

$$\Psi_4 = \sum_{i=1,12} \psi(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$$

The i-amplitude has total JJ_z and total isospin TT_z

$$\psi(\mathbf{x}_i, \mathbf{y}_i) = \sum_{\alpha=1}^{N_c} \phi_\alpha(x_i, y_i) \mathcal{Y}_\alpha(i, j, k)$$

$$\psi(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i) = \sum_{\alpha=1}^{N_c} \phi_\alpha(x_i, y_i, z_i) \mathcal{Y}_\alpha(i, j, k, m)$$

The angular-spin-isospin basis elements are

$$\mathcal{Y}_\alpha(i, j, k) = \left\{ \left[Y_{l_{1\alpha}}(\hat{x}_i) Y_{l_{2\alpha}}(\hat{y}_i) \right]_{\Lambda_\alpha} \left[s_\alpha^{ij} s_\alpha^k \right]_{S_\alpha} \right\}_{JJ_z} \left[t_\alpha^{jk} t_\alpha^i \right]_{TT_z}$$

$$\mathcal{Y}_\alpha(i, j, k, m) = \left\{ \left[Y_{l_{1\alpha}}(\hat{x}_i) Y_{l_{2\alpha}}(\hat{y}_i) Y_{l_{3\alpha}}(\hat{z}_i) \right]_{\Lambda_\alpha} \left[[s_\alpha^{ij} s_\alpha^k]_{S_{1\alpha}} s_\alpha^m \right]_{S_{2\alpha}} \cdots \right.$$

Jacobi Coordinates:

$A = 3$

$\mathbf{x}_i = \mathbf{r}_j - \mathbf{r}_k$

$\mathbf{y}_i = \sqrt{\frac{4}{3}}(\frac{\mathbf{r}_j + \mathbf{r}_k}{2} - \mathbf{r}_i)$

$A = 4$

$\mathbf{x}_i = \mathbf{r}_j - \mathbf{r}_k$

$\mathbf{y}_i = \sqrt{\frac{4}{3}}(\frac{\mathbf{r}_j + \mathbf{r}_k}{2} - \mathbf{r}_i)$

$\mathbf{z}_i = \sqrt{\frac{3}{2}}(\frac{\mathbf{r}_j + \mathbf{r}_k + \mathbf{r}_i}{3} - \mathbf{r}_m)$

Hyperspherical Variables:

$x_i = \rho \cos\theta_i$

$y_i = \rho \sin\theta_i$

$[\rho, \Omega_i] = [\rho, \theta_i, \hat{x}_i, \hat{y}_i]$

$x_i = \rho \cos\theta_{1i}$

$y_i = \rho \sin\theta_{1i} \cos\theta_{2i}$

$z_i = \rho \sin\theta_{1i} \sin\theta_{2i}$

$[\rho, \Omega_i] = [\rho, \theta_{1i}, \theta_{2i}, \hat{x}_i, \hat{y}_i, \hat{z}_i]$

Using this coordinates the Hamiltonian is

$$H = \frac{-\hbar^2}{2m} \sum_i \nabla^2_{r_i} + \sum_{i<j} V(i,j) + \sum_{i<j<k} W(i,j,k)$$

$$H - T_{cm} = \frac{-\hbar^2}{m} \sum_i \nabla^2_i + \sum_{i<j} V(i,j) + \sum_{i<j<k} W(i,j,k)$$

$$T = \frac{-\hbar^2}{m} \sum_i \nabla^2_i = \frac{-\hbar^2}{m} [L^2_\rho + \frac{L^2(\Omega)}{\rho^2}]$$

HH basis

The $A = 3$ amplitudes are expanded in the (P)HH basis

$$\phi_\alpha(x_i, y_i) = \rho^{-5/2} f_\alpha(x_i) \left[\sum_K u_K^\alpha(\rho)\ {}^{(2)}P_K^{l_{1\alpha}, l_{2\alpha}}(\theta_i) \right]$$

The grand angular quantum number is $K = l_{1\alpha} + l_{2\alpha} + 2n$

The HH basis element is

$$\mathcal{Y}_{[K]}^\alpha = [f_\alpha(x_i)]^{(2)} P_K^{l_{1\alpha}, l_{2\alpha}}(\theta_i) \mathcal{Y}_\alpha(i, j, k)$$

The $A = 4$ amplitudes are expanded in the HH basis

$$\phi_\alpha(x_i, y_i, z_i) = \rho^{-4} \left[\sum_{[K]} u_{[K]}^\alpha(\rho)\ {}^{(3)}P_{[K]}^{l_{1\alpha}, l_{2\alpha}, l_{3\alpha}}(\theta_{1i}, \theta_{2i}) \right]$$

$K = l_{1\alpha} + l_{2\alpha} + l_{3\alpha} + 2n_1 + 2n_2$

$$\mathcal{Y}_{[K]}^\alpha = {}^{(3)}P_{[K]}^{l_{1\alpha}, l_{2\alpha}, l_{3\alpha}}(\theta_{1i}, \theta_{2i}) \mathcal{Y}_\alpha(i, j, k, m)$$

$$L^2(\Omega) \mathcal{Y}_{[K]}^\alpha = K(K + 4) \mathcal{Y}_{[K]}^\alpha$$

$$L^2(\Omega) \mathcal{Y}_{[K]}^\alpha = K(K + 7) \mathcal{Y}_{[K]}^\alpha$$

Symmetrization

$$\Psi_3 = \sum_{i=1,3} \psi(\mathbf{x}_i, \mathbf{y}_i)$$

$$\Psi_3 = \sum_{i=1,3} \left[\sum_{\alpha=1}^{N_c} \phi_\alpha(x_i, y_i) \mathcal{Y}_\alpha(i,j,k) \right]$$

$$\Psi_3 = \rho^{-5/2} \sum_{i=1,3} \left[\sum_{\alpha=1}^{N_c} \sum_{[K]} u^\alpha_{[K]}(\rho) \mathcal{Y}^\alpha_{[K]}(i,j,k) \right]$$

$$\Psi_3 = \rho^{-5/2} \left[\sum_{\alpha=1}^{N_c} \sum_{[K]} u^\alpha_{[K]}(\rho) \sum_{i=1,3} \mathcal{Y}^\alpha_{[K]}(i,j,k) \right]$$

Antisymmetric HH basis element

$$B^\alpha_{[K]}(\Omega) = \sum_{i=1,3} \mathcal{Y}^\alpha_{[K]}(i,j,k)$$

$l_{1\alpha} + s^\alpha_{ij} + t^\alpha_{ij}$ = odd, parity =$(-1)^{l_{1\alpha}+l_{2\alpha}}$

$$\Psi_3 = \rho^{-5/2} \left[\sum_{\alpha=1}^{N_c} \sum_{[K]} u^\alpha_{[K]}(\rho) B^\alpha_{[K]}(\Omega) \right]$$

Symmetrization

$$\Psi_4 = \sum_{i=1,12} \psi(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$$

$$\Psi_4 = \sum_{i=1,12} \left[\sum_{\alpha=1}^{N_c} \phi_\alpha(x_i, y_i, z_i) \mathcal{Y}_\alpha(i,j,k,m) \right]$$

$$\Psi_4 = \rho^{-4} \sum_{i=1,12} \left[\sum_{\alpha=1}^{N_c} \sum_{[K]} u^\alpha_{[K]}(\rho) \mathcal{Y}^\alpha_{[K]}(i,j,k,m) \right]$$

$$\Psi_4 = \rho^{-4} \left[\sum_{\alpha=1}^{N_c} \sum_{[K]} u^\alpha_{[K]}(\rho) \sum_{i=1,12} \mathcal{Y}^\alpha_{[K]}(i,j,k,m) \right]$$

Antisymmetric HH basis element

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$$\Psi_4 = \rho^{-4} \left[\sum_{\alpha=1}^{N_c} \sum_{[K]} u^\alpha_{[K]}(\rho) B^\alpha_{[K]}(\Omega) \right]$$

K	$L=0$		$L=1$		$L=2$	
	M_{KL}	M'_{KL}	M_{KL}	M'_{KL}	M_{KL}	M'_{KL}
0	2	1				
2	10	1	9	1	6	1
4	30	4	45	4	30	3
6	70	8	135	12	89	9
8	140	14	315	27	205	18
10	252	24	630	54	405	36
12	420	41	1,134	96	721	63
14	660	59	1,890	160	1,190	102
16	990	90	2,970	250	1,854	158
18	1,430	128	4,455	375	2,760	236
20	2,002	176	6,435	488	3,960	321
22	2,730	235	9,009	585	5,511	385
24	3,640	282	12,285	675	7,475	445
30	3,876		9,180		16,540	
40	10,626		26,565		47,145	
50	23,751		61,425		107,900	

Antisymmetrical HH-spin-isospin states for $J=0, T=0$ and $\pi=+$. M_{KL} is the total number of the states. M'_{KL} is the number of the linearly independent states with $\ell_1+\ell_2+\ell_3 \leq 6$.

α-particle calculation:

1. Class C1. HH sates with T=0 and $l_2 = l_3 = 0$ and $n_2 = 0$. There are three channels of this kind. This is also the most slowly convergent class. States up to $K_1 = 72$ have been included.

2. Class C2. HH sates with T=0 and $l_2 = l_3 = 0$ but $n_2 > 0$. States up to $K_1 = 40$ have been included.

3. Class C3. Remaining HH $T = 0$ states of the channels having $\ell_1 + \ell_2 + \ell_3 = 2$. There are 20 more possible channels. $K_3 \approx 30$ included in the expansion.

4. Class C4. HH $T = 0$ states belonging to the channels with $\ell_1 + \ell_2 + \ell_3 = 4$. There are 57 channels of this kind. $K_4 \approx 24$ has been considered.

5. Class C5. HH $T = 0$ states belonging to the channels with $\ell_1 + \ell_2 + \ell_3 = 6$. There are 109 channels of this kind. $K_5 \approx 20$ has been considered.

6. Class C6. This class includes the states having $T > 0$. We have included in the expansion all the channels of this kind with $\ell_1 + \ell_2 + \ell_3 \leq 2$ (45 channels). $K_6 \approx 16$ has been considered.

The final calculation includes about 8000 HH states

$A = 3,4$ basis elements

The hyperradial functions can be expanded as

$$u^{\alpha}_{[K]} = \sum_m A^{\alpha}_{m[K]} \mathcal{L}^{(n)}_m(\beta\rho)\ \mathrm{e}^{-\beta\rho}$$

with β a nonlinear parameter and $n = 5, 8$ for $A = 3, 4$.

Then

$$\Psi_{A=3,4} = \sum_{\alpha,m,[K]} A^{\alpha}_{m[K]} |\alpha\, m\, [K] >$$

$|\alpha\, m\, [K] >$ is a complete antisymmetric basis element.

The bound state is obtained from the generalized problem

$$\sum_{\alpha,m,[K]} A^{\alpha}_{m[K]} < \alpha'\, m'\, [K'] | H - E | \alpha\, m\, [K] >= 0$$

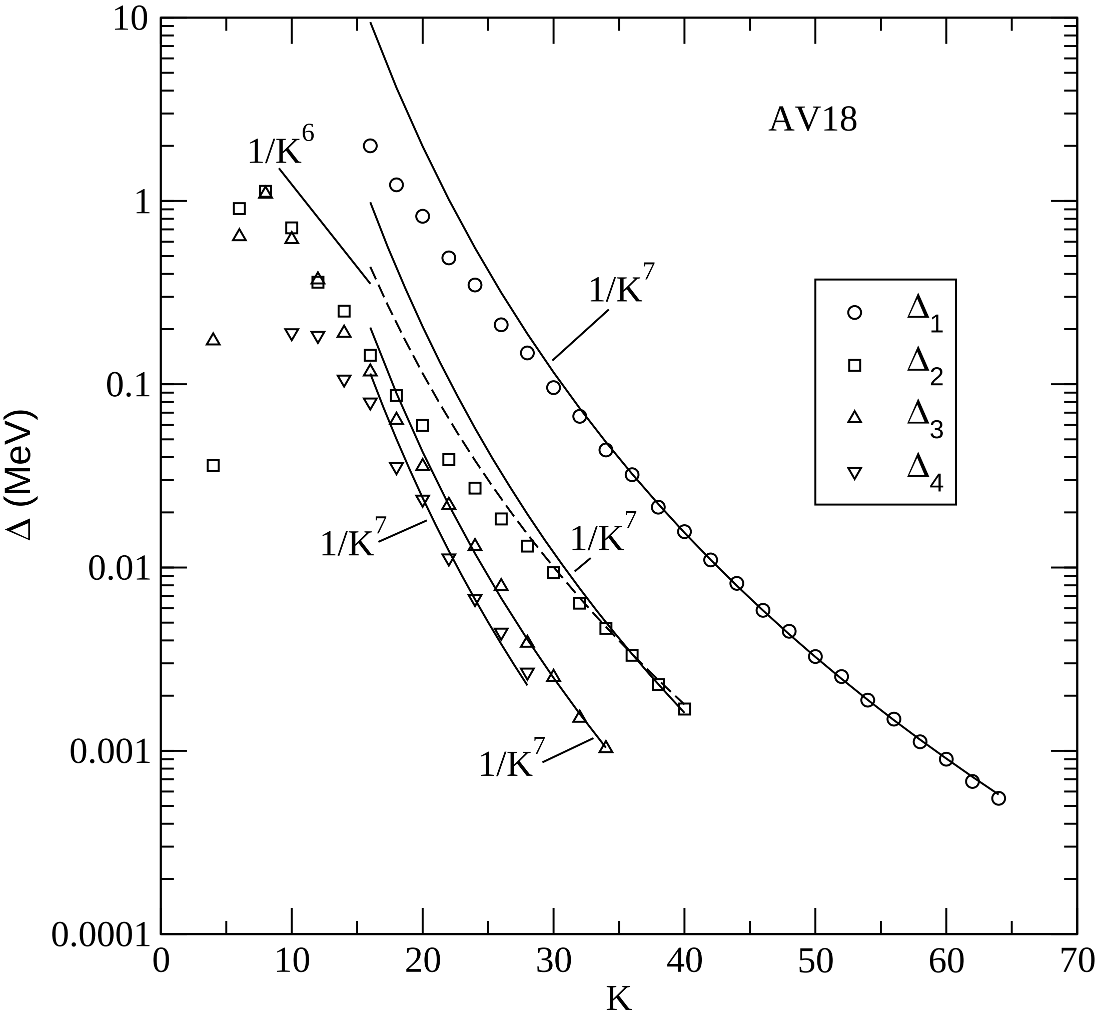
Moreover

$$< \rho, \Omega | \alpha\, m\, [K] >= \mathcal{L}^{(n)}_m(\beta\rho)\ \mathrm{e}^{-\beta\rho} B^{\alpha}_{[K]}(\Omega)$$

$$< P, \Omega_p | \alpha\, m\, [K] >= \mathcal{F}^{(n)}_m(\beta P) B^{\alpha}_{[K]}(\Omega_p)$$

The calculations can be done in CS or MS

K_1	K_2	K_3	K_4	K_5	K_6	MT-V	AV18	AV18+UIX
20						28.928	14.70	14.90
30						29.794	15.99	16.17
40						29.962	16.17	16.34
50						30.008	16.20	16.377
60						30.024	16.213	16.385
70						30.032	16.214	16.386
72						30.033	16.214	16.386
72	8					30.714	18.29	18.99
72	16					31.170	19.75	20.65
72	24					31.240	19.97	20.87
72	32					31.256	20.01	20.912
72	36					31.259	20.022	20.918
72	40					31.261	20.026	20.921
72	40	8				31.300	21.94	24.69
72	40	16				31.336	23.24	27.15
72	40	24				31.340	23.37	27.35
72	40	30				31.341	23.385	27.37
72	40	34				31.341	23.388	27.37
72	40	34	8			31.341	23.54	27.55
72	40	34	16			31.344	24.086	28.31
72	40	34	20			31.346	24.145	28.38
72	40	34	24			31.347	24.163	28.40
72	40	34	28			31.347	24.170	28.41
72	40	34	28	16			24.181	28.43
72	40	34	28	20			24.191	28.439
72	40	34	28	24			24.195	28.444
72	40	34	28	24	4		24.205	28.456
72	40	34	28	24	8		24.209	28.461
72	40	34	28	24	12		24.210	28.462
72	40	34	28	24	16		24.210	28.462
“extrapolated”						31.358	24.23	28.47
“exact”						31.360	24.25	28.50



Convergence studies: binding energy differences for the classes C1 – C4 as function of the grand angular value K.

$$\Delta_1(K) = B(K,0,0,0,0,0) - B(K-2,0,0,0,0,0)$$
$$\Delta_2(K) = B(K_{1\mathrm{M}},K,0,0,0,0) - B(K_{1\mathrm{M}},K-2,0,0,0,0)$$
$$\Delta_3(K) = B(K_{1\mathrm{M}},K_{2\mathrm{M}},K,0,0,0) - B(K_{1\mathrm{M}},K_{2\mathrm{M}},K-2,0,0,0)$$

$K_{1\mathrm{M}} = 72$
$K_{2\mathrm{M}} = 40$
$K_{3\mathrm{M}} = 34$

The HH Method applied to the A=4 bound state

M. Viviani, A. Kievsky, S. Rosati, PRC71, 024006 (2005)

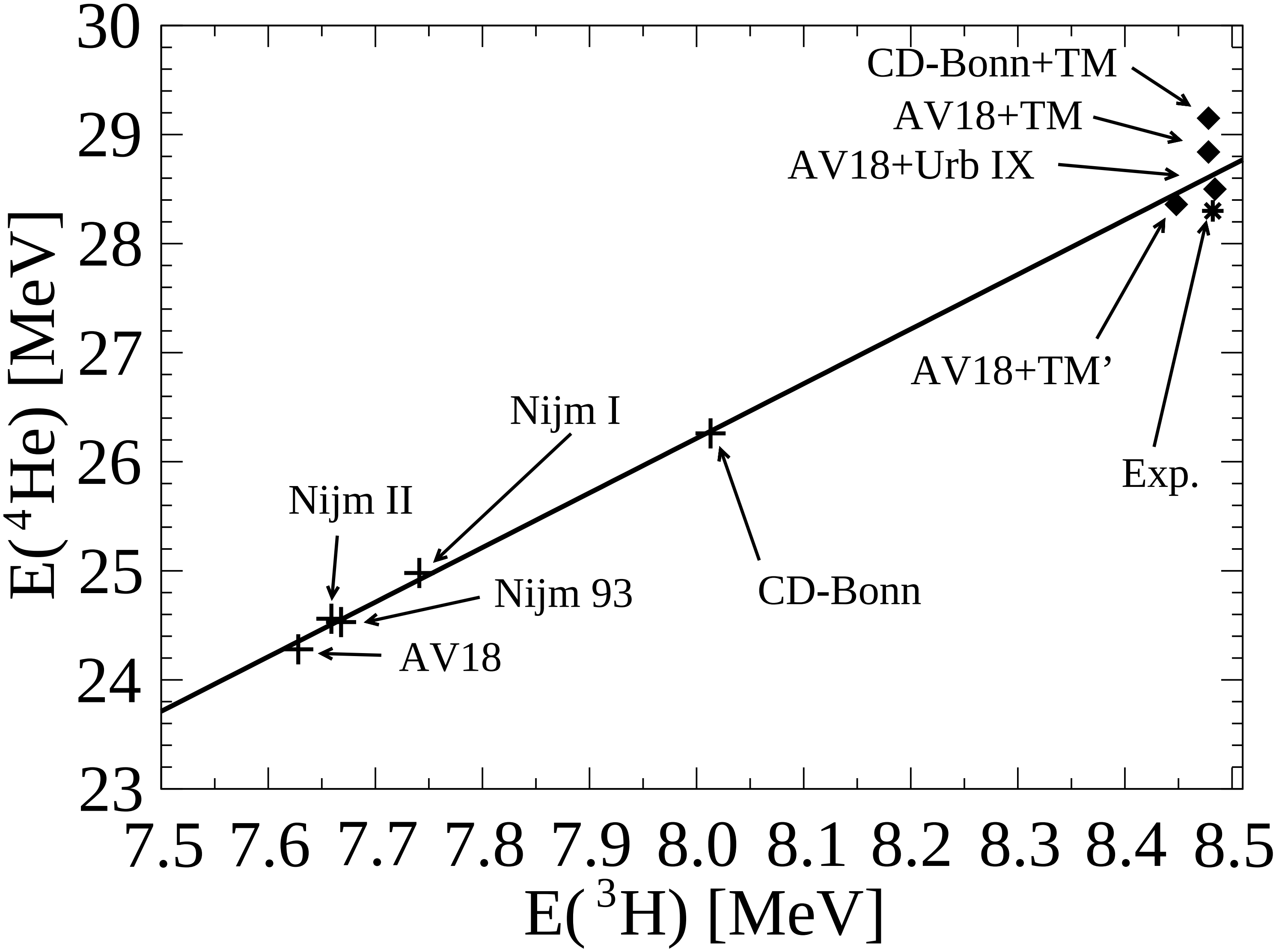
potential	Method	B(MeV)	T(MeV)	P_P(%)	P_D(%)
AV18	HH	24.22	97.84	0.35	13.74
	FY(MS)	24.25	97.80	0.35	13.78
	FY(CS)	24.22	97.77		
Nijm II	HH	24.43	100.27	0.33	13.37
	FY(CS)	24.56	100.31	0.33	13.37
AV18+UR	HH	28.46	113.30	0.73	16.03
	FY(MS)	28.50	113.21	0.75	16.03
	GFMC	28.34(4)	110.7(7)		
AV18+TM'	HH	28.31	110.27	0.73	15.63
	FY(MS)	28.36	110.14	0.75	15.67
Exp.		28.30			
CD-Bonn	HH	26.23	78.58	0.228	10.51
	FY(MS)	26.26	77.15	0.22	10.72
N3LO	HH	25.09	68.79	0.172	8.951
	FY(MS)	25.41			
	NCSM	25.36			
$V(low-k)$	HH	27.95	61.51	0.116	7.031

Binding Energies of $A = 3$

Hamiltonian	^{3}H	^{3}He
	B(MeV)	B(MeV)
AV18 ($T = 1/2$)	7.618	6.917
AV18 ($T = 1/2, 3/2$)	7.624	6.925
AV18+UR ($T = 1/2$)	8.474	7.742
AV18+UR ($T = 1/2, 3/2$)	8.479	7.750
Expt.	8.482	7.718

Interaction term	$B(^3$H)-$B(^3$He)
Nuclear CSB	65 keV
Point Coulomb	677 keV
Full Coulomb	648 keV
Magnetic moment	17 keV
Orbit-orbit force	7 keV
n-p mass difference	14 keV
Total (theory)	751 keV
Expt.	764 keV

A. Nogga,...,A.K, M.V., L.M.,S.R, Phys.Rev.C 67, 034004 (2003)



CD-Bonn+TM
AV18+TM
AV18+Urb IX
AV18+TM’
Exp.
Nijm I
Nijm II
Nijm 93
AV18
CD-Bonn
E(⁴He) [MeV]
E(³H) [MeV]
30
29
28
27
26
25
24
23
7.5
7.6
7.7
7.8
7.9
8.0
8.1
8.2
8.3
8.4
8.5

Asymptotic Constants

$$< Y_L(\hat{r})|\Psi_d(\mathbf{r}) > \to C_L^{np} f_L^{np} \frac{\mathrm{e}^{-k_d r}}{r} \qquad L = 0, 2$$

$$< Y_L(\hat{y})\Psi_d^S(\mathbf{x})|\Psi_t(\mathbf{x},\mathbf{y}) > \to C_L^{nd} f_L^{nd} \frac{\mathrm{e}^{-k_d r_{nd}}}{r_{nd}} \qquad L = 0, 2$$

$$< Y_L(\hat{y})\Psi_d^S(\mathbf{x})|\Psi_h(\mathbf{x},\mathbf{y}) > \to C_L^{pd} f_L^{pd} \frac{W_{-\eta,i_L}(r_{pd})}{r_{pd}} \qquad L = 0, 2$$

$$< Y_0(\hat{z})\Psi_t^S(\mathbf{x},\mathbf{y})|\Psi_4(\mathbf{x},\mathbf{y},\mathbf{z}) > \to C_0^{pt} f_0^{pt} \frac{W_{-\eta,1/2}(r_{pt})}{r_{pt}}$$

$$< Y_0(\hat{z})\Psi_h^S(\mathbf{x},\mathbf{y})|\Psi_4(\mathbf{x},\mathbf{y},\mathbf{z}) > \to C_0^{nh} f_0^{nh} \frac{\mathrm{e}^{-k_h r_{nh}}}{r_{nh}}$$

$$< \Psi_d(\mathbf{x})Y_L(\hat{y})\Psi_d(\mathbf{z})|\Psi_4(\mathbf{x},\mathbf{y},\mathbf{z}) > \to C_L^{dd} f_L^{dd} \frac{W_{-\eta,i_L}(r_{dd})}{r_{dd}}$$

$$L = 0, 2$$

Parameter	C_S^{pd}	C_D^{pd}	C_S^{nd}	C_D^{nd}	η
AV18UR	1.878	0.0752			0.0400
Ayer *et al.*					0.0386 ± 0.0045
AV18UR			1.854	0.0798	0.0430
Kozlowska *et al.*					0.0411 ± 0.0025
George *et al.*					0.0431 ± 0.0025

ANC's and the parameter η for He and H

Parameter	C_S^{pt}	C_S^{nh}	C_S^{dd}	C_D^{dd}	D_2^{dd} (fm^2)
AV18	1.72	1.67	1.96	-0.209	-0.115
AV18UR	1.75	1.69	1.99	-0.277	-0.113
Adhikari *et al.*					-0.12
Karp *et al.*					-0.3 ± 0.1
Merz *et al.*					-0.19 ± 0.04
Weller *et al.*					-0.2 ± 0.05

ANC's and the parameter D_2^{dd} for the α particle

$A = 3, 4$ scattering states

The A=3,4 scattering wave function is written as

$$\Psi = \Psi_C + \Psi_A$$

The internal part is

$$\Psi_C = \sum_{i=1,3} \psi(\mathbf{x}_i, \mathbf{y}_i)$$

$$\Psi_C = \sum_{i=1,12} \psi(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$$

The i-amplitude has total JJ_z and total isospin TT_z and are expanded in terms of the (P)HH basis.

The second term, Ψ_A describes the asymptotic motion of bound state relative to the incident nucleon. It can be written as a sum of three amplitudes with the generic one having the form

$$\Omega^\lambda_{LSJ}(\mathbf{x}_i, \mathbf{y}_i) = \mathcal{R}^\lambda_L(y_i) \left\{ [\phi_d(\mathbf{x}_i) s^i]_S Y_L(\hat{y}_i) \right\}_{JJ_z} [t_d^{jk} t^i]_{TT_z}$$

$$\Omega^\lambda_{LSJ}(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i) = \mathcal{R}^\lambda_L(z_i) \left\{ [\phi_3(\mathbf{x}_i, \mathbf{y}_i) s^m]_S Y_L(\hat{z}_i) \right\}_{JJ_z} [t_3^{ijk} t^m]_{TT_z}$$

The functions $\mathcal{R}^\lambda$ are related to the regular or irregular Coulomb (spherical Bessel) functions. The functions Ω^λ can be combined to form a general asymptotic state ${}^{(2S+1)}L_J$

$$\Omega^+_{LSJ} = \Omega^0_{LSJ} + \sum_{L'S'} {}^J\mathcal{S}^{SS'}_{LL'} \Omega^1_{L'S'J}$$

The $A = 3, 4$ scattering w.f. for an incident state with relative angular momentum L, spin S and total angular momentum J has the form

$$\Psi^{+}_{LSJ} = \sum_{p} \left[\Psi_C(p) + \Omega^{+}_{LSJ}(p)\right]$$

A variational estimate of the trial parameters in the w.f. Ψ^{+}_{LSJ} can be obtained using the Kohn Variational Principle

$$[{}^J\mathcal{S}^{SS'}_{LL'}] = {}^J\mathcal{S}^{SS'}_{LL'} - \mathrm{i}\langle\Psi^{-}_{LSJ}|H - E|\Psi^{+}_{L'S'J}\rangle$$

The observables can be calculated from the transition matrix M which can be decomposed as a sum of the Coulomb amplitude f_c plus a nuclear term

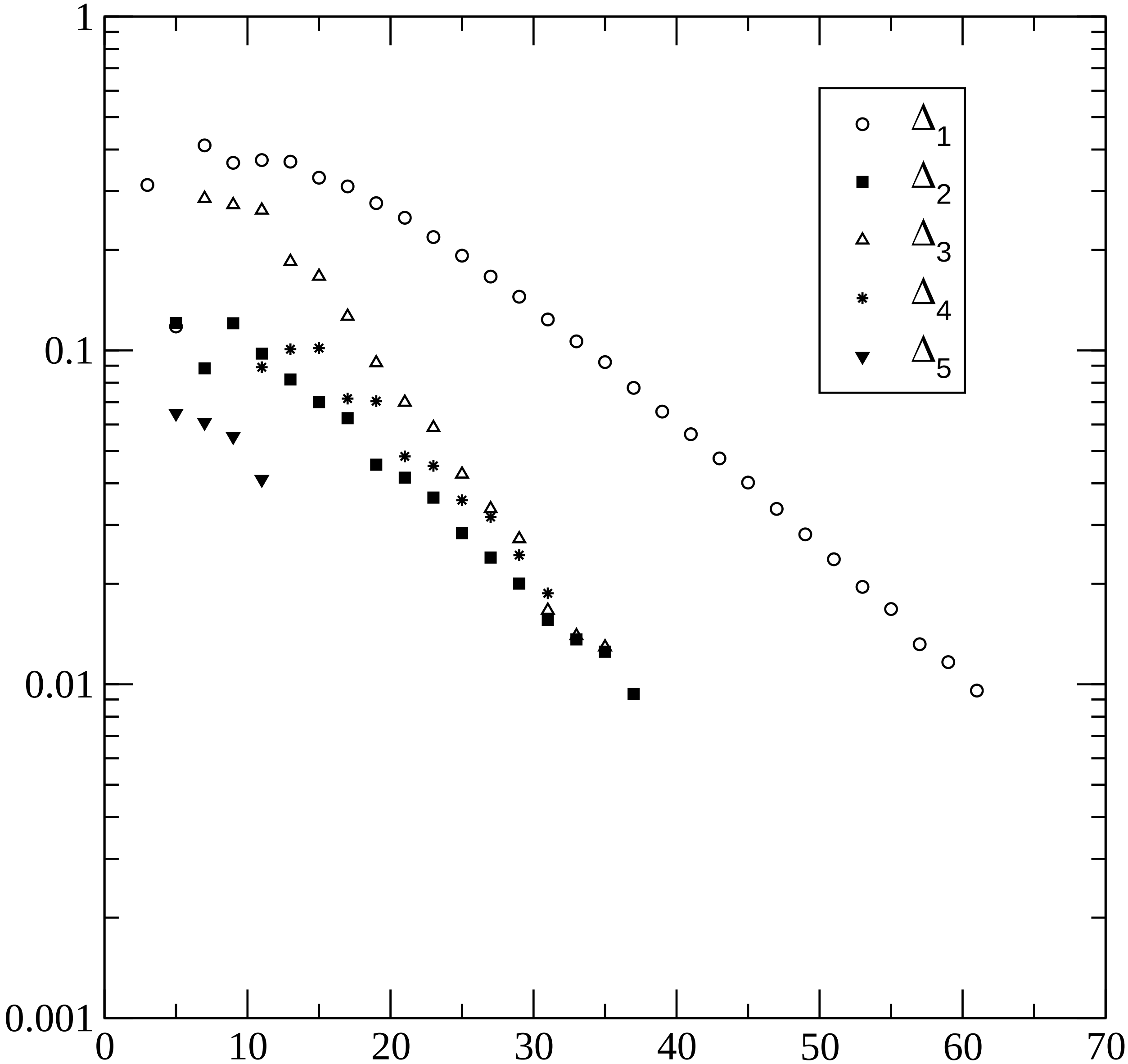
$$M^{SS'}_{\nu\nu'}(\theta) = f_c(\theta)\delta_{SS'}\delta_{\nu\nu'} + \frac{\sqrt{4\pi}}{k} \sum_{L,L',J} \sqrt{2L+1}(L0S\nu|J\nu)$$

$$(L'M'S'\nu'|J\nu) \exp[i(\sigma_L + \sigma_{L'} - 2\sigma_0)] \; {}^J T^{SS'}_{LL'} \; Y_{L'M'}(\theta, 0)$$

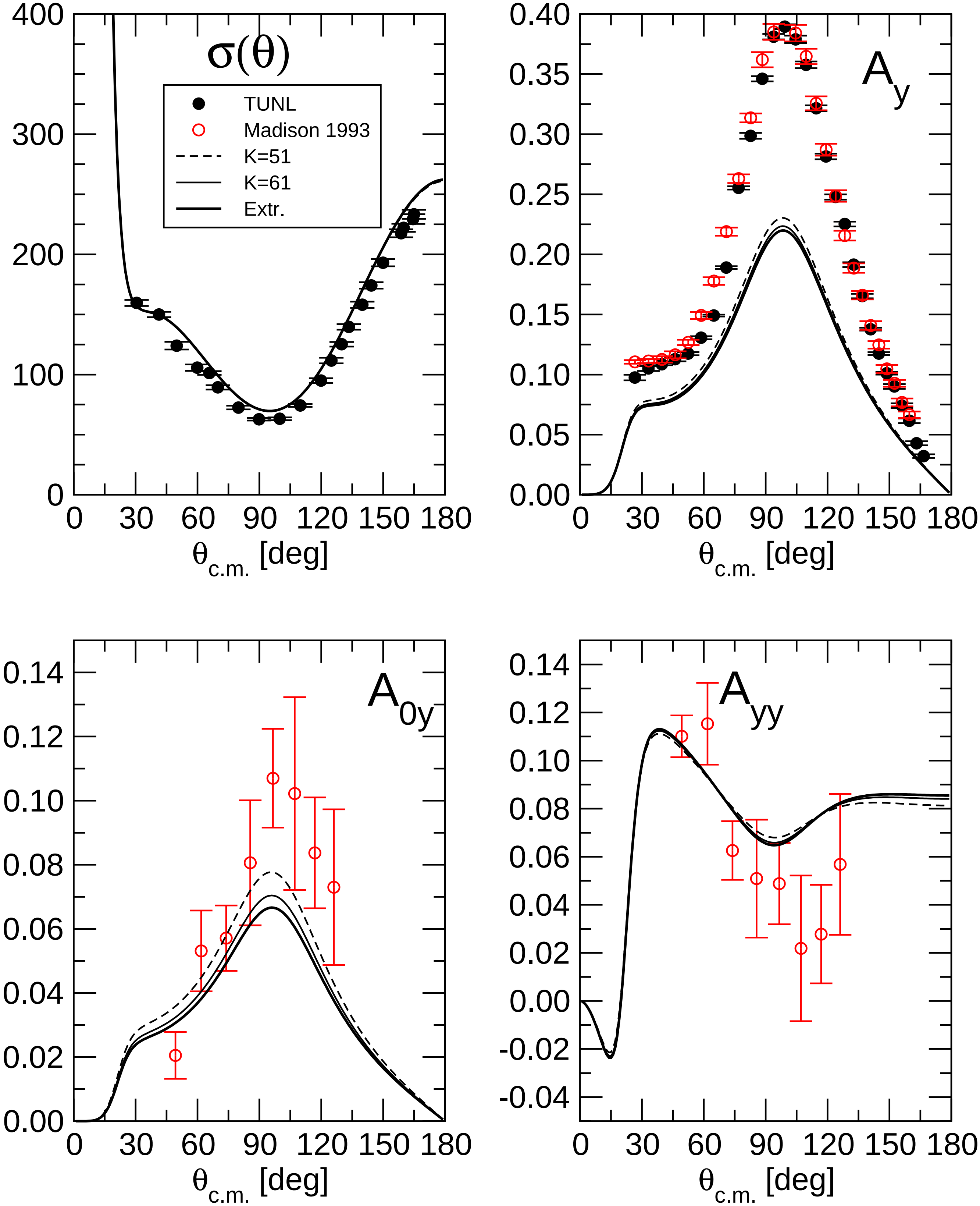
$\sigma = \frac{tr\{MM^\dagger\}}{6}$	$\sigma = \frac{tr\{MM^\dagger\}}{4}$
$A_y = \frac{tr\{M\sigma_y M^\dagger\}}{6}$	$A_{y0} = \frac{tr\{M\sigma_y M^\dagger\}}{4}$
$iT_{11} = \frac{tr\{MS_y M^\dagger\}}{6}$	$A_{0y} = \frac{tr\{MS_y M^\dagger\}}{4}$
.	.
.	.

K_1	K_2	K_3	K_4	K_5	K_6	K_7	η	δ (deg)
21							1.00197	10.721
31							1.00160	11.567
41							1.00124	11.965
51							1.00103	12.138
61							1.00092	12.209
61	11						1.00004	12.634
61	21						1.00002	12.935
61	31						1.00000	13.060
61	35						1.00000	13.086
61	35	11					1.00001	15.328
61	35	21					1.00000	15.968
61	35	31					1.00000	16.151
61	35	35					1.00000	16.178
61	35	35	11				1.00001	16.317
61	35	35	15				1.00000	16.519
61	35	35	21				1.00000	16.709
61	35	35	25				1.00000	16.746

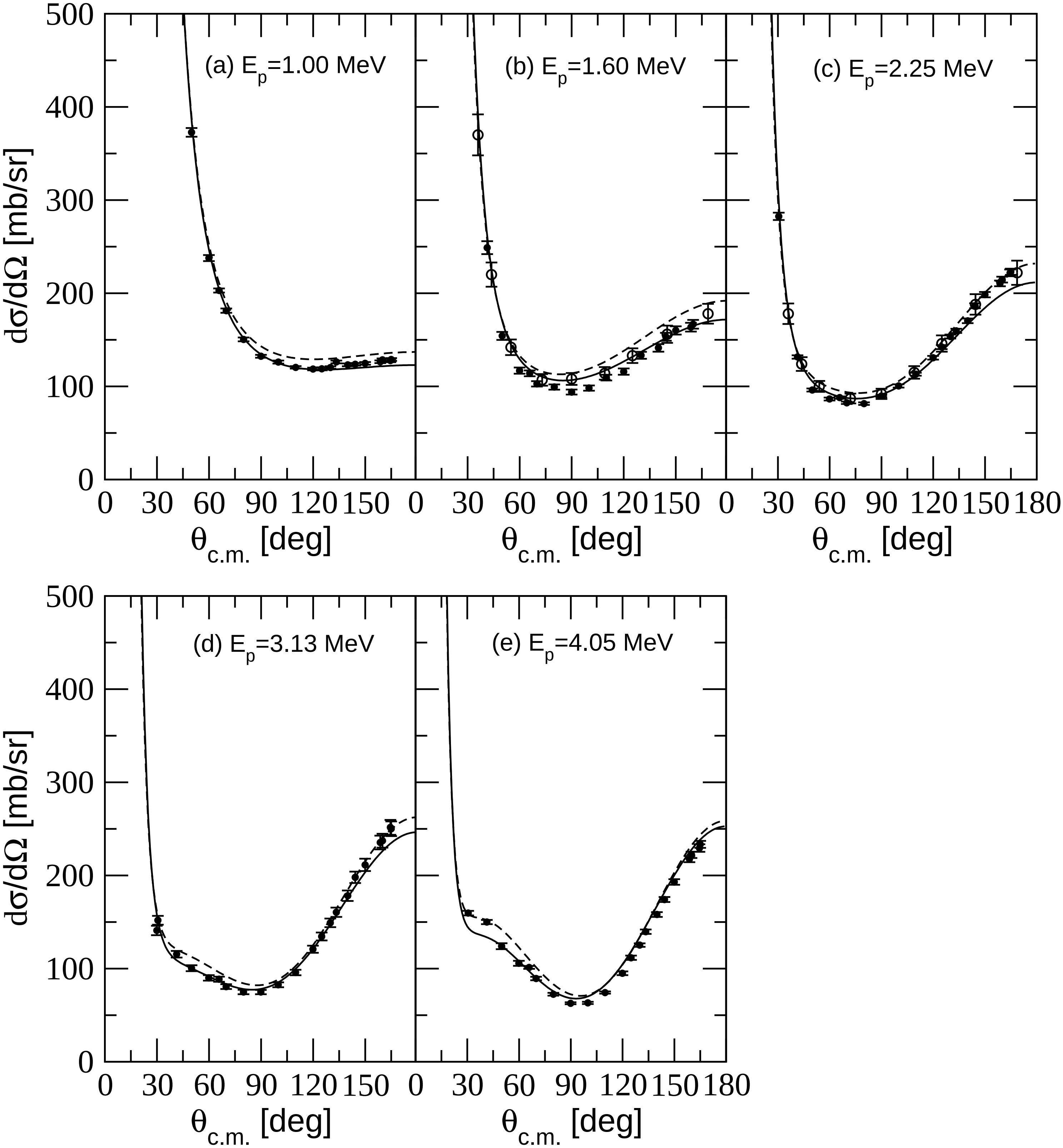
Convergence of 0^- inelasticity parameter and phase-shift (deg) at $E_p = 4.05$ MeV for $p - {}^3$He scattering. The S-matrix is $S = \eta \exp(2i\delta)$. The AV18 potential is considered here with the inclusion of the point Coulomb interaction.



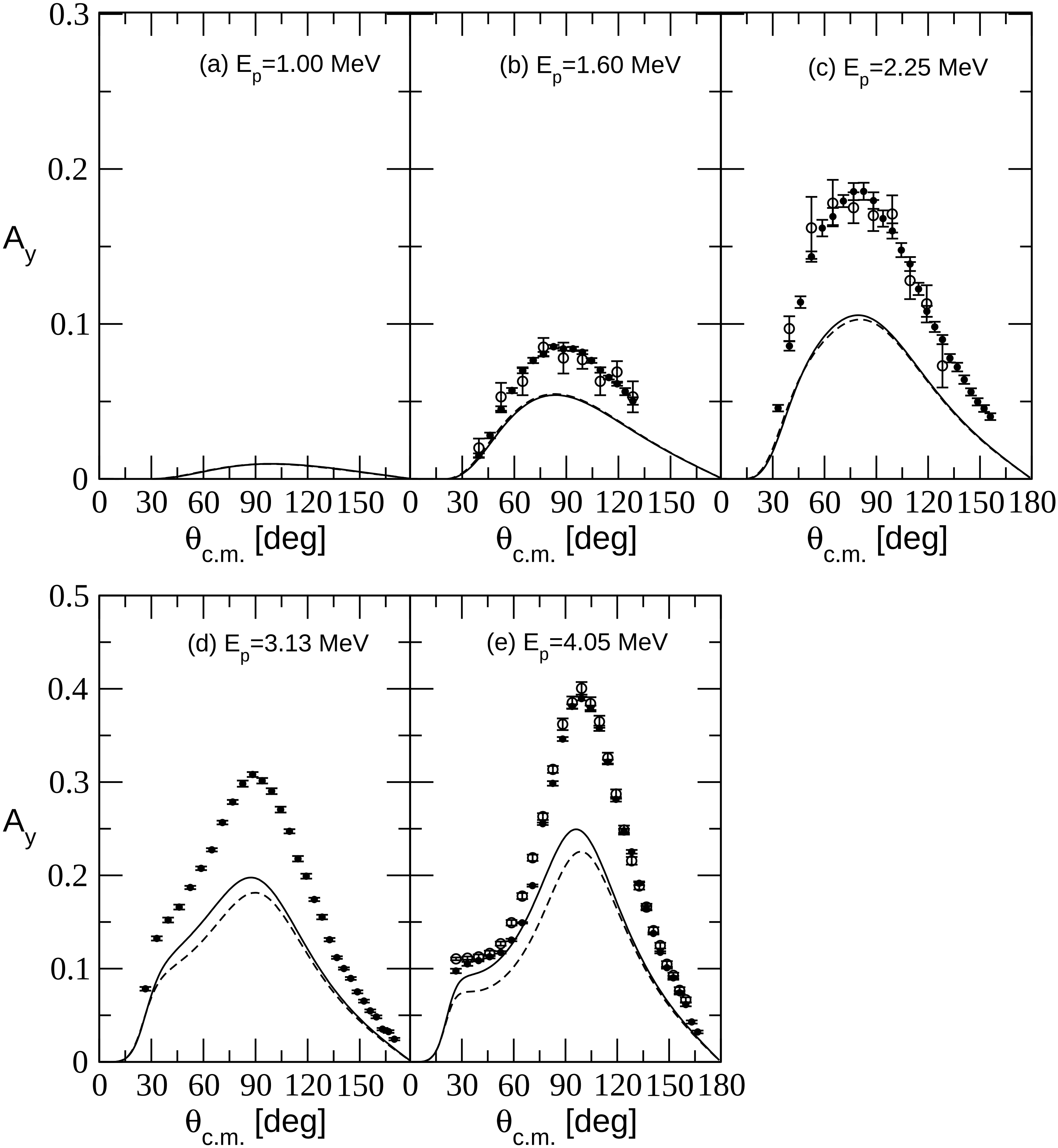
$$\begin{aligned}
\Delta_1(K) &= \delta(K,0,0,0,0,0,0) - B(K-2,0,0,0,0,0,0)\ , \\
\Delta_2(K) &= \delta(\overline{K_1},K,0,0,0,0,0) - \delta(\overline{K_1},K-2,0,0,0,0,0)\ , \qquad \overline{K} \\
\Delta_3(K) &= \delta(\overline{K_1},\overline{K_2},K,0,0,0,0) - \delta(\overline{K_1},\overline{K_2},K-2,0,0,0,0)\ ,
\end{aligned}$$



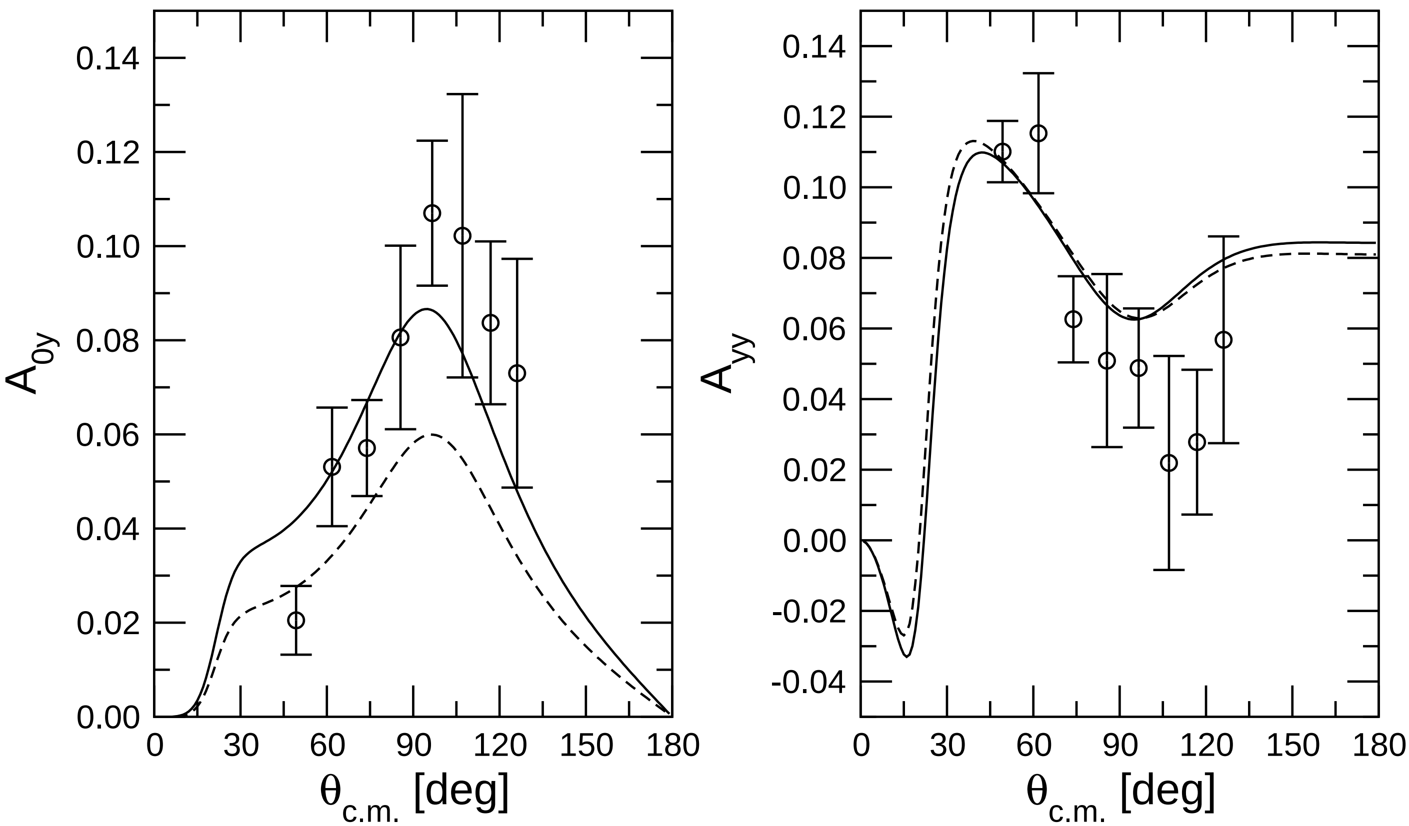
Four $p - {}^3$He elastic scattering observables at $E_p = 4.05$ MeV calculated using different values for the 0^- phase-shift (AV18 potential)



The measured $p -^3$ He elastic differential cross sections at five different energies are compared with the data and the calculations for the AV18 (dashed lines) and AV18/UIX (solid lines)

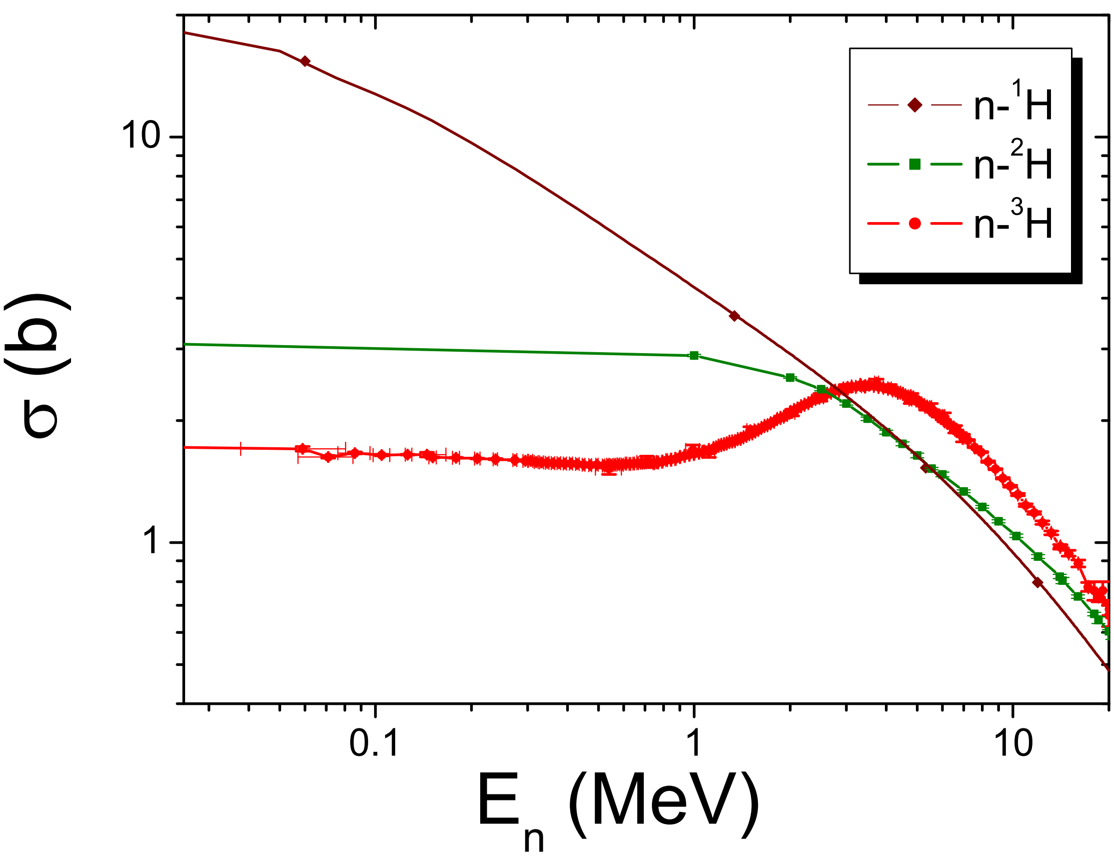


The measured $p-{}^3$He proton analyzing power. The solid circles are the data from TUNL, open circles from Madison. Calculations are for the AV18 (dashed lines) and AV18/UIX (solid lines)

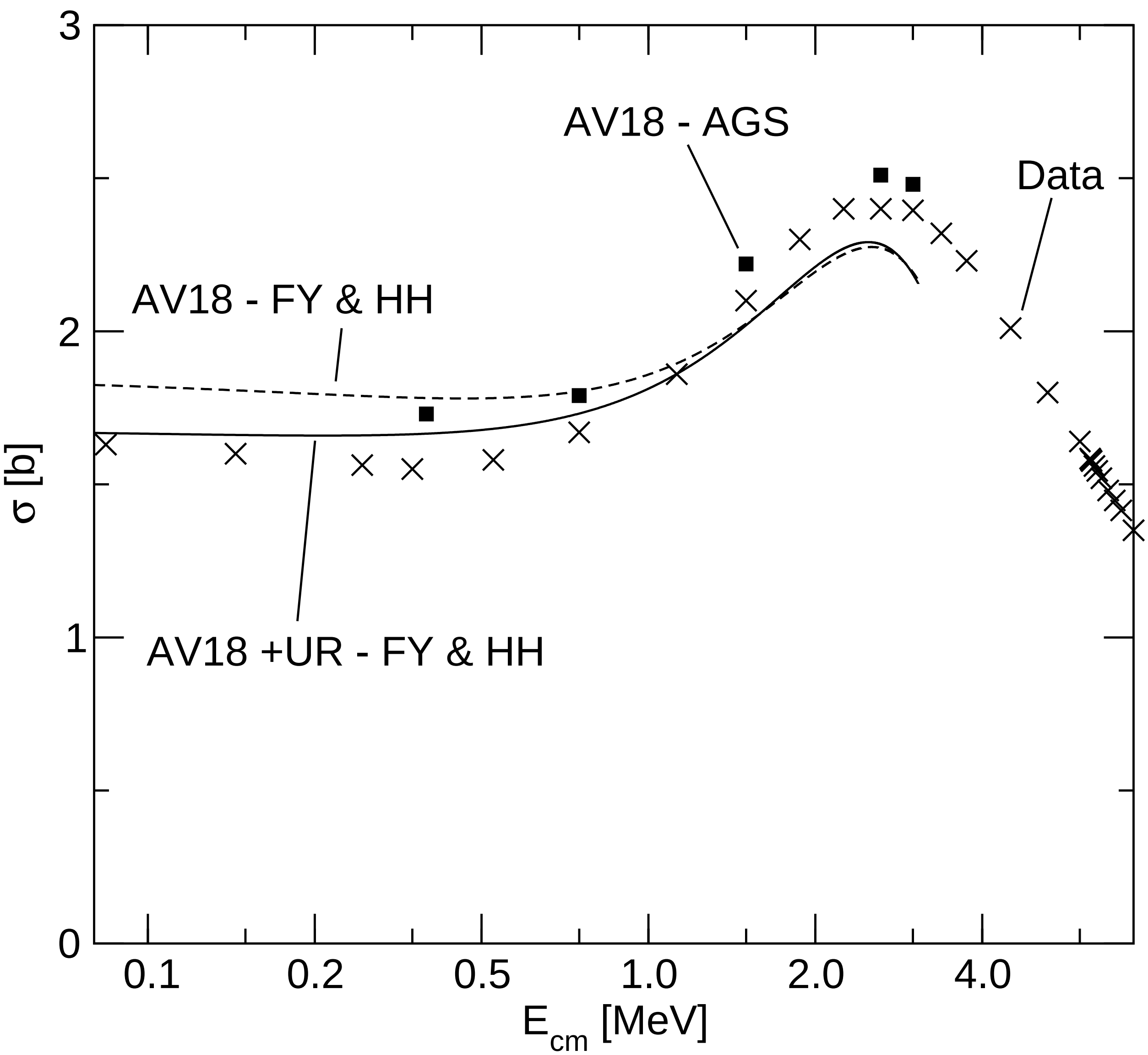


The p-^{3}He analyzing power A_{0y} and A_{yy} at $E_p = 4.05$ MeV. Data are from Madison. The curves show the results of the theoretical calculations for the AV18 (dashed lines) and AV18/UIX (solid lines) potential models

$n - ^3$H **total cross section**



$n-{}^{3}\mathrm{H}$ total cross section



Lazauskas et al. PRC 71, 034004 (2005)

$A=3$ continuum with Coulomb

The A=3 scattering wave function is written as

$$\Psi = \Psi_C + \Psi_A$$

The internal part is

$$\Psi_C = \sum_{i=1,3} \psi(\mathbf{x}_i, \mathbf{y}_i)$$

The i-amplitude are expanded in terms of the (P)HH basis.

The second term, Ψ_A describes the asymptotic motion of the deuteron relative to the incident proton.

$$\Omega^{\lambda}_{LSJ}(\mathbf{x}_i, \mathbf{y}_i) = \mathcal{R}^{\lambda}_{L}(y_i) \left\{ [\phi_d(\mathbf{x}_i) s^i]_S Y_L(\hat{y}_i) \right\}_{JJ_z} [t^{jk}_d t^i]_{TT_z}$$

The functions $\mathcal{R}^{\lambda}$ are related to the regular or irregular Coulomb functions. The functions Ω^{λ} can be combined to form a general asymptotic state $^{(2S+1)}L_J$

$$\Omega^{+}_{LSJ} = \Omega^{0}_{LSJ} + \sum_{L'S'} {}^J\mathcal{S}^{SS'}_{LL'} \Omega^{1}_{L'S'J}$$

Ω^{+}_{LSJ} is a solution of the asymptotic Hamiltonian

$$H_0 = T + V(1,2) + \frac{e^2}{r_{pp}}$$

Above the breakup threshold the channel with three outgoing nucleons is open. This new asymptotic configuration is describe by the hyperradial functions.

$$u_\mu(\rho) \to -\sum_{\mu'} \left(e^{-i\hat{\chi}\,\ln 2Q\rho}\right)_{\mu\mu'} \, b_{\mu'} \, e^{iQ\rho}$$

where $b_{\mu'}$ are unknown coefficients. The dimensionless operator $\hat{\chi}$ originates from the Coulomb interaction as

$$\hat{\chi} = \frac{M_N}{2\hbar^2 Q} \sum_{i=1}^{3} \frac{e^2}{\cos\phi_i} \frac{1+\tau_{j,z}}{2} \frac{1+\tau_{k,z}}{2}$$

In this way the functions $u_\mu(\rho)$ are outgoing solutions of the asymotitc hamiltonian

$$H_0 = T + \frac{e^2}{r_{pp}}$$

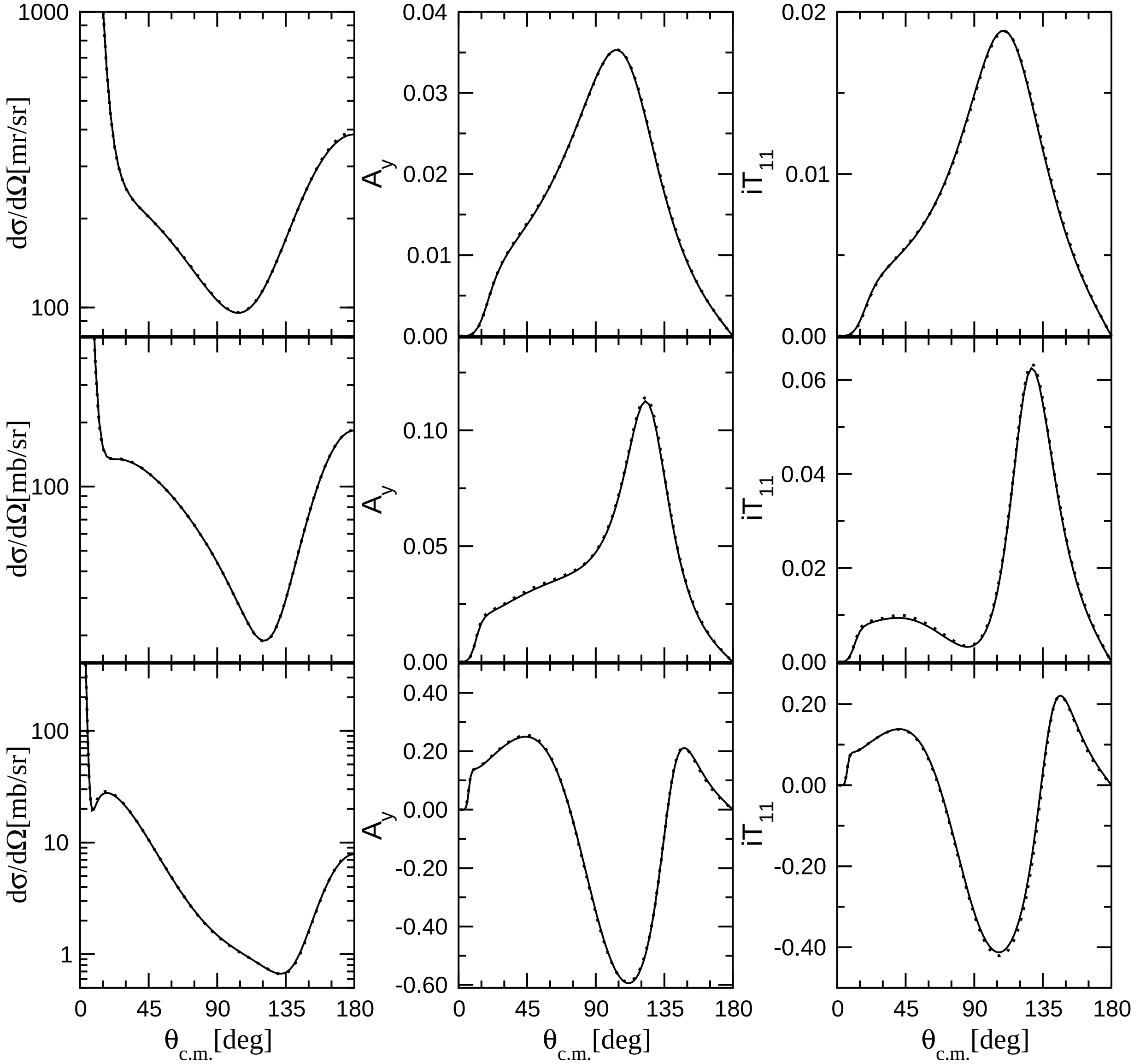
In a different approach the Coulomb interaction can be taken into account in momentum space via the solution of the Alt-Grassberger-Sandhas equation using the screened Coulomb potential and the renormalization approach.

Deltuva et al. PRC, 054005 (2005) , Deltuva et al. PRC, 064003 (2005) p-d benchmark below breakup Pisa-Los Alamos PRC63, 064004 (2001)

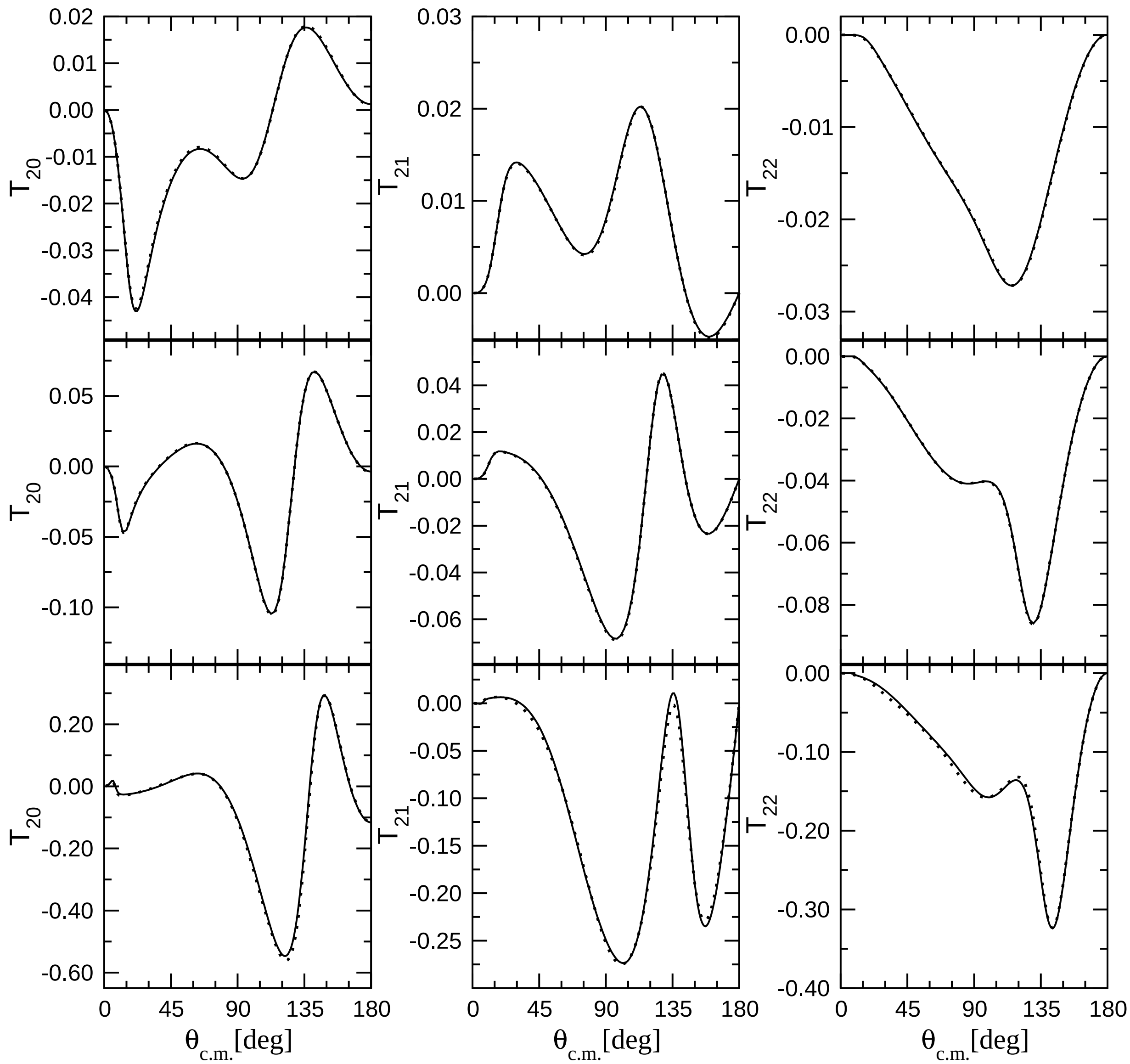
n-d benchmark above breakup Bochum-Los Alamos PRC51, 2356 (1995)

	${}^2\delta_0$	${}^2\eta_0$	${}^4\delta_0$	${}^4\eta_0$
nd at 14.1 MeV	105.48	0.4649	68.95	0.9782
	105.49	0.4649	68.95	0.9782
nd at 42.0 MeV	41.34	0.5022	37.72	0.9033
	41.36	0.5022	37.71	0.9033
pd at 14.1 MeV	108.44	0.4984	72.60	0.9795
	108.39	0.4983	72.62	0.9796
pd at 42.0 MeV	43.67	0.5056	39.95	0.9046
	43.70	0.5056	39.97	0.9046

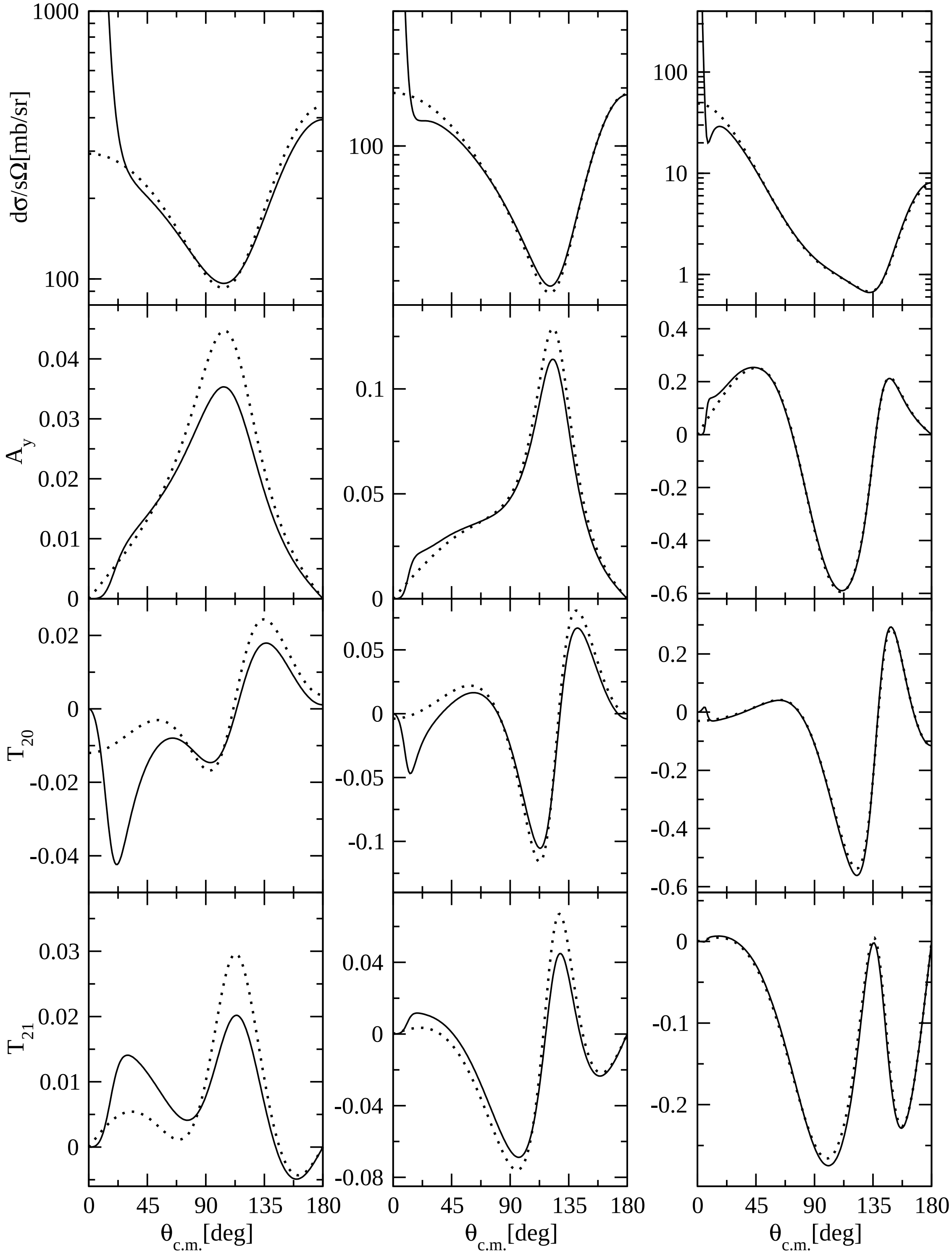
nd and pd phase-shift and inelasticity parameters calculated with MT I-III potential. The KVP and integral equation approach results are given in the first and second row, respectively



σ, A_y and iT_{11} for pd scattering at 3 MeV (first row), 10 MeV (second row) and 65 MeV (third row). The KVP (thin solid line) and IEA (dotted line) are compared.



T_{20}, T_{21}, and T_{22} for pd scattering at 3 MeV (first row), 10 MeV (second row) and 65 MeV (third row). The KVP (thin solid line) and IEA (dotted line) are compared.



pd (solid line) and nd (dotted line) are compared for σ, A_y, T_{20} and T_{21} at 3 MeV (first column), 10 MeV (middle column) and 65 MeV (last column).

$A = 3$ continuum beyond Coulomb

The long range part of the Magnetic Moment interaction is

$$v_{MM}(pp) = -\frac{\alpha}{4M_p^2}[\mu_p^2\frac{S_{ij}}{r^3} + (8\mu_p - 2)\frac{\mathbf{L}\cdot\mathbf{S}}{r^3}]$$

$$v_{MM}(np) = -\frac{\alpha\mu_n}{4M_nM_p}[\mu_p\frac{S_{ij}}{r^3} + \frac{M_p}{2M_r}\frac{(\mathbf{L}\cdot\mathbf{S}+\mathbf{L}\cdot\mathbf{A})}{r^3}]$$

$$v_{MM}(nn) = -\frac{\alpha}{4M_n^2}\mu_n^2\frac{S_{ij}}{r^3}$$

Scattering observables can be described through the transition matrix

$$M^{SS'}_{\nu\nu'}(\theta) = f_c(\theta)\delta_{SS'}\delta\nu\nu'$$
$$+\frac{\sqrt{4\pi}}{k}\sum_{LL'J}^{L_{max}}\sqrt{2L+1}(L0S\nu|J\nu)(L'M'S'\nu'|J\nu)$$
$$\times\ \exp[i(\sigma_L+\sigma_{L'}-2\sigma_0)]\ {}^JT^{SS'}_{LL'}\ Y_{L'M'}(\theta,0)$$

with the Coulomb amplitude

$$f_c(\theta) = \sum_{L=0}^{\infty}(2L+1)(\mathrm{e}^{2i\sigma_L}-1)P_L(\cos\theta)$$
$$= -2i\eta\frac{\mathrm{e}^{2i\sigma_0}}{1-\cos\theta}\mathrm{e}^{-i\eta\ln(\frac{1-\cos\theta}{2})}$$

N-d MM interaction

Summing the corresponding $V^{MM}(NN)$ for a nucleon far from the deuteron or calculating the spin$\frac{1}{2}\otimes$spin1 one-photon exchange diagram:

$$V_{nd}^{MM} = -\frac{\alpha}{r^3}[\frac{\mu_n\mu_d}{M_n M_d} S_{nd}^I + \frac{\mu_n}{2M_n M_{nd}}(\mathbf{L}\cdot\mathbf{S}_{nd} + \mathbf{L}\cdot\mathbf{A}_{nd})]$$

$$\begin{aligned} V_{pd}^{MM} &= -\frac{\alpha}{r^3}[\frac{\mu_p\mu_d}{M_p M_d} S_{pd}^I - \frac{Q_d}{2} S_d^{II} \\ &+(\frac{\mu_p}{2M_n M_{pd}} - \frac{1}{4M_p^2})(\mathbf{L}\cdot\mathbf{S}_{pd} + \mathbf{L}\cdot\mathbf{A}_{pd}) \\ &+(\frac{\mu_d}{2M_d M_{pd}} - \frac{1}{4M_d^2})(\mathbf{L}\cdot\mathbf{S}_{pd} - \mathbf{L}\cdot\mathbf{A}_{pd})] \end{aligned}$$

$$S_{Nd}^I = 3(\mathbf{S}_N\cdot\hat{r})(\mathbf{S}_d\cdot\hat{r}) - \mathbf{S}_N\cdot\mathbf{S}_d, \quad N = n, p$$
$$S_d^{II} = 3(\mathbf{S}_d\cdot\hat{r})^2 - 2$$
$$\mathbf{S}_{Nd} = \mathbf{S}_N + \mathbf{S}_d$$
$$\mathbf{A}_{Nd} = \mathbf{S}_N - \mathbf{S}_d.$$

Transition matrix with MM interaction

Including the spin-orbit MM terms, the transition matrix is

$$M_{\nu\nu'}^{SS'}(\theta) = f_c(\theta)\delta_{SS'}\delta_{\nu\nu'} + f_{so}(\theta)$$
$$+\frac{\sqrt{4\pi}}{k}\sum_{LL'J}^{L_{max}}\sqrt{2L+1}(L0S\nu|J\nu)(L'M'S'\nu'|J\nu)$$
$$\times\ \exp[i(\sigma_L+\sigma_{L'}-2\sigma_0)]\ {}^JT_{LL'}^{SS'}\ Y_{L'M'}(\theta,0)$$

with the spin-orbit amplitude

$$f_{so}(\theta) = f_{\mu\mu'}^{SS'}\left[\frac{\cos\theta + 2\mathrm{e}^{-i\eta\ln(\frac{1-\cos\theta}{2})} - 1}{\sin\theta}\right.$$
$$\left. - \sum_{L=1}^{L_{max}}\frac{(2L+1)}{L(L+1)}\mathrm{e}^{2i(\sigma_L-\sigma_0)}P_L^1(\cos\theta)\right]$$

This final form have been derived from

$$\sum_{L=1}^{\infty}\frac{(2L+1)}{L(L+1)}\mathrm{e}^{2i\sigma_L}P_L^1(\cos\theta) = \frac{\mathrm{e}^{2i\sigma_0}}{\sin\theta}$$
$$\times[\cos\theta + 2\mathrm{e}^{-i\eta\ln(\frac{1-\cos\theta}{2})} - 1]$$

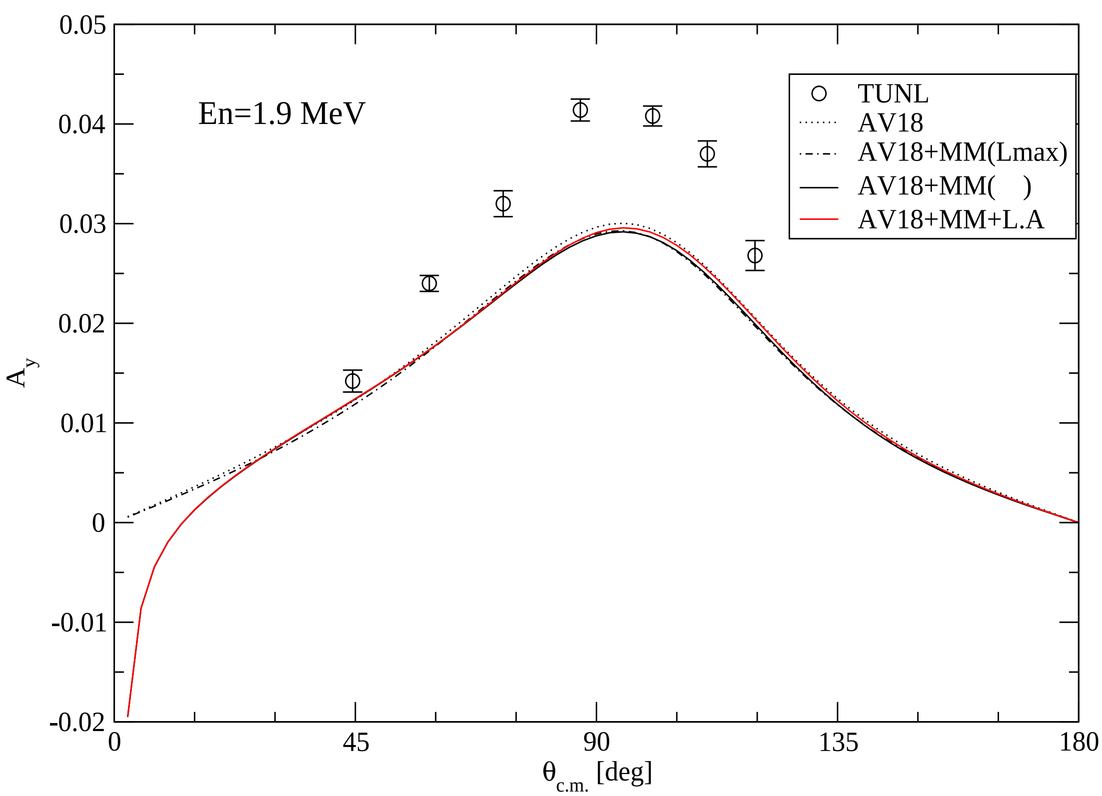
A.Kievsky, M. Viviani, L. Marcucci, PRC69, 014002 (2004)

n-d case

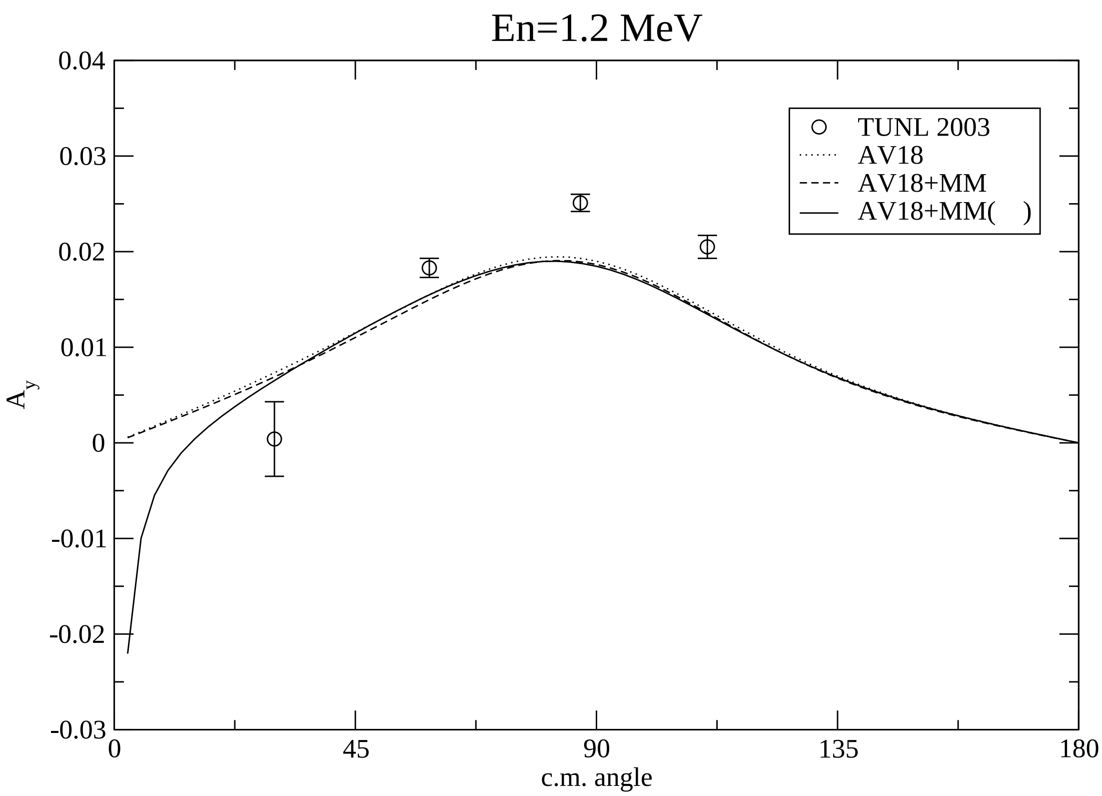
$$M_{\nu\nu'}^{SS'}(\theta) = f_{so}(\theta) + \frac{\sqrt{4\pi}}{k} \sum_{LL'J}^{L_{max}} \sqrt{2L+1}$$
$$\times (L0S\nu|J\nu)(L'M'S'\nu'|J\nu)\, {}^{J}T_{LL'}^{SS'}\, Y_{L'M'}(\theta,0)$$

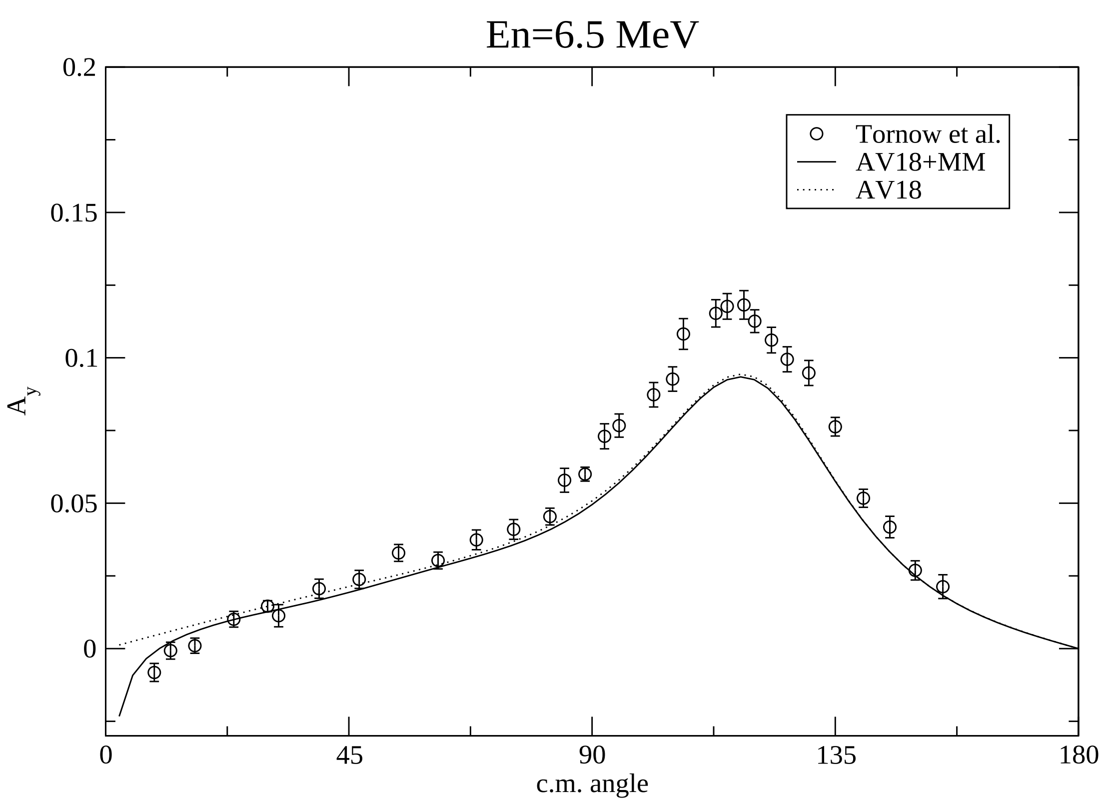
with the spin-orbit amplitude

$$f_{so}(\theta) = f_{\mu\mu'}^{SS'} \left[\frac{\cos\theta + 1}{\sin\theta} - \sum_{L=1}^{L_{max}} \frac{(2L+1)}{L(L+1)} P_L^1(\cos\theta) \right]$$



$$A_y = \frac{\sigma(\uparrow) - \sigma(\downarrow)}{\sigma(\uparrow) + \sigma(\downarrow)}$$



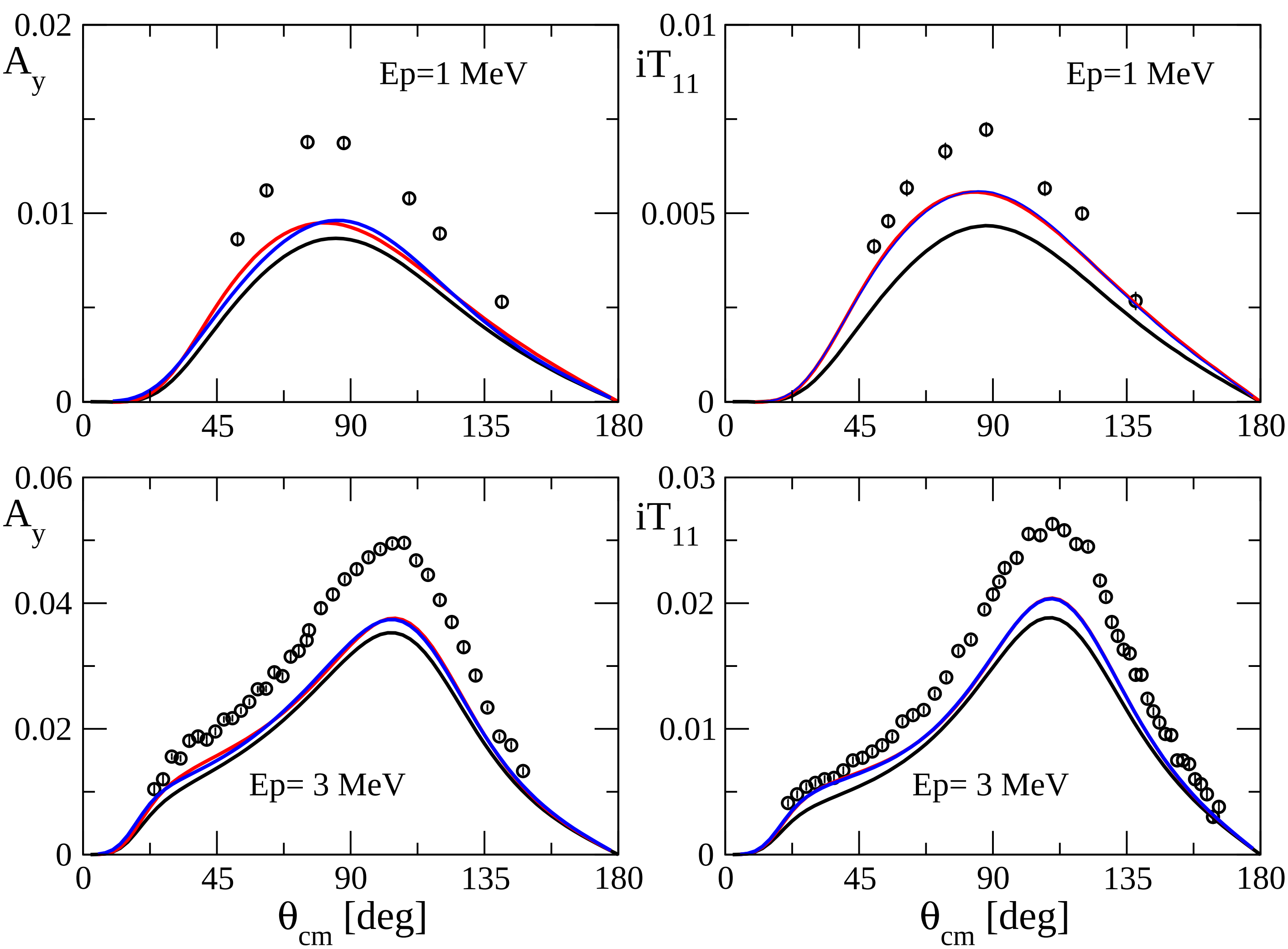


p-d scattering

E_p =1 MeV ○ M. Wood et al. (TUNL)

E_p =3 MeV ○ K. Sagara et al.

— AV18 - - - - AV18+MM — AV18+MM(∞)

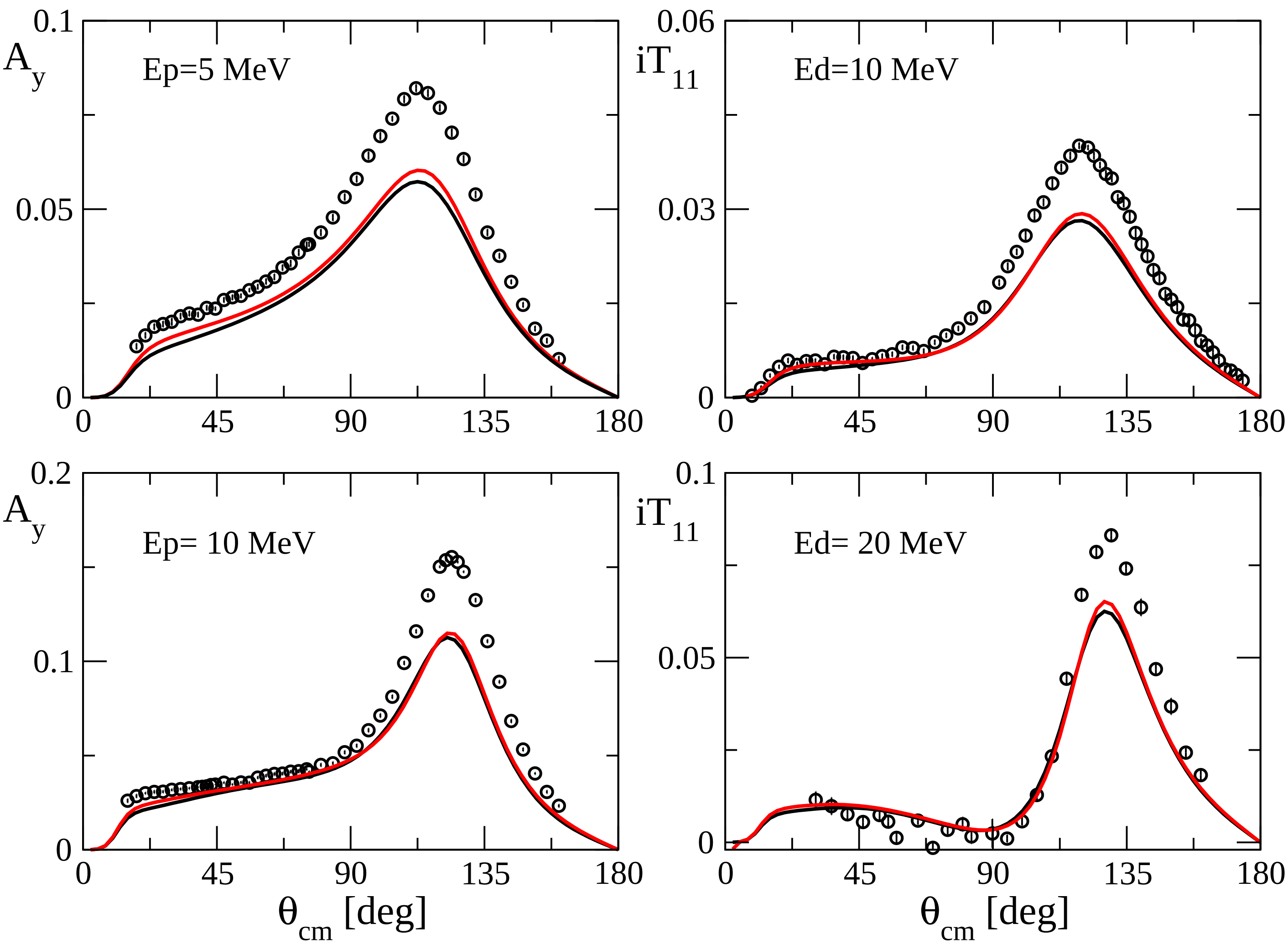


p-d scattering

$E_p = 5, 10$ MeV, $E_d = 10$ MeV ○ K. Sagara et al.

$E_d = 20$ MeV ○ W. Grüebler et al.

— AV18 - - - - AV18+MM



Conclusions

- Benchmarks are important to establish the methods and the models
- Detailed wave functions have been constructed using the HH basis for bound states in $A = 3, 4$
- The HH basis can be used in CS or in MS as well
- Different models predict a different structure of the bound state: the ANC's can be related to observables
- Benchmarks have been performed for $n -^3$ H in A=4 and for $p - d$ scattering in $A = 3$
- The MM interaction has been included in the Hamiltonian
- Extension of the HH technique to $A > 4$